ANNIHILATOR OF LOCAL COHOMOLOGY MODULES AND STRUCTURE OF RINGS

Tran Nguyen An
TNU - University of Education

ABSTRACT

Let (R, m) be a Noetherian local ring, A an Artinian R-module, and M a finitely generated R-module. It is clear that Ann R(M/ p M) = p, for all p $\in$ Var(Ann R M). Therefore, it is natural to consider the following dual property for annihilator of Artinian modules:

Ann R(0 : A p) = p, for all p $\in$ Var(Ann R A). ($*$)

Let $i \geq 0$ be an integer. Alexander Grothendieck showed that the local cohomology module Hmi (M) of M is Artinian. The property ($*$) of local cohomology modules is closed related to the structure of the base ring. In this paper, we prove that for each p $\in$ Spec(R) such that Hmi (R/ p) satisfies the property (*) for all i, then R/ p is universally catenary and the formal fibre of R over p is Cohen-Macaulay.

Keywords: *Local cohomology; universally catenary; formal fibre; Artinian module; CohenMacaulay ring*

Received: 26/5/2020; Revised: 29/8/2020; Published: 04/9/2020

LINH HÓA TỬ CỦA MÔĐUN ĐỐI ĐỒNG ĐIỀU ĐỊA PHƯƠNG VÀ CẤU TRÚC VÀNH

Trần Nguyên An
Trường Đại học Sư phạm - ĐH Thái Nguyên

TÓM TẮT

Cho (R, m) là vành Noether địa phương, A là R-môđun Artin, và M là R-môđun hữu hạn sinh. Ta có Ann R(M/ p M) = p với mọi p $\in$ Var(Ann R M). Do đó rất tự nhiên ta xét tính chất sau về linh hóa tử của môđun Artin

Ann R(0 : A p) = p for all p $\in$ Var(Ann R A). ($*$)

Cho $i \geq 0$ là số nguyên. Alexander Grothendieck đã chỉ ra rằng môđun đối đồng điều địa phương Hi m(M) là Artin. Tính chất ($*$) của các môđun đối đồng điều địa phương liên hệ mật thiết với cấu trúc vành cơ sở. Trong bài báo này, chúng tôi chỉ ra với mỗi p $\in$ Spec(R) mà Hmi (R/ p) thỏa mãn tính chất (*) với mọi i thì R/ p là catenary phổ dụng và các thớ hình thức của R trên p là Cohen-Macaulay.

Từ khóa: *Đối đồng điều địa phương; catenary phổ dụng; thớ hình thức; môđun Artin; vành Cohen-Macaulay*

Ngày nhận bài: 26/5/2020; Ngày hoàn thiện: 29/8/2020; Ngày đăng: 04/9/2020

Email: antn@tnue.edu.vn
https://doi.org/10.34238/tnu-jst.3194

1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring, A an Artinian R-module, and M a finitely generated R-module of dimension d. For each ideal I of R, we denote by $\mathrm{Var}(I)$ the set of all prime ideals containing I. For a subset T of $\mathrm{Spec}(R)$, we denote by $\min(T)$ the set of all minimal elements of T under the inclusion.

It is clear that $\mathrm{Ann}_R(M/\mathfrak{p}M) = \mathfrak{p}$, for all $\mathfrak{p} \in \mathrm{Var}(\mathrm{Ann}_R M)$. Therefore, it is natural to consider the following dual property for annihilator of Artinian modules:

$$Ann_R(0 :_A \mathfrak{p}) = \mathfrak{p}, \forall \mathfrak{p} \in \mathrm{Var}(\mathrm{Ann}_R A).(*)$$

If R is complete with respect to $\mathfrak{m}$-adic topology, it follows by Matlis duality that the property (*) is satisfied for all Artinian R-modules. However, there are Artinian modules which do not satisfy this property. For example, by [1, Example 4.4], the Artinian R-module $H^1_{\mathfrak{m}}(R)$ does not satisfy the property (*), where R is the Noetherian local domain of dimension 2 constructed by M. Ferrand and D. Raynaud [2] (see also [3, App. Ex. 2] Ex. 2]) such that its $\mathfrak{m}$-adic completion $\widehat{\mathrm{R}}$ has an associated prime $\mathfrak{q}$ of dimension 1. In [4], N. T. Cuong, L. T. Nhan and N. T. Dung showed that the top local cohomology module $H^d_{\mathfrak{m}}(M)$ satisfies property $(*)$ if and only if the ring $R/\mathrm{Ann}_R(M/U_M(0))$ is catenary, where $U_M(0)$ is the largest submodule of M of dimension less than d. The property $(*)$ of local cohomology modules is closed related to the structure of the ring. In [5], L. T. Nhan and the author proved that if $H^i_{\mathfrak{m}}(M)$ satisfies the property (*) for all i , then $R/\mathfrak{p}$ is unmixed for all $\mathfrak{p} \in \mathrm{Ass}\, M$ and the ring $R/\mathrm{Ann}_R M$ is universally catenary. The following conjecture was given by N. T. Cuong in his seminar.

Conjecture 1.1. *The following statements are equivalent:*

(i) $H^i_{\mathfrak{m}}(R)$ satisfies the property (*) for all i;

(ii) R is universally catenary and all its formal flbers are Cohen-Macaulay.

L. T. Nhan and T. D. M. Chau proved in [6] that $H^i_{\mathfrak{m}}(M)$ satisfies the property (*) for all i, for all finitely generated R-module M if and only if R is universally catenary and all its formal flbers are Cohen-Macaulay. The following result is the main result of this paper. We hope that we can use this to give a positive answer for the above conjecture.

Theorem 1.2. *Assume $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $H^i_{\mathfrak{m}}(R/\mathfrak{p})$ satisfies the property (*) for all i. Then $R/\mathfrak{p}$ is universally catenary and the formal fibre of R over $\mathfrak{p}$ is Cohen-Macaulay.*

2. Proof of the main results

The theory of secondary representation was introduced by I. G. Macdonald (see [7]) which is in some sense dual to that of primary decomposition for Noetherian modules. Note that every Artinian R-module A has a minimal secondary representation $A = A_1 + \ldots + A_n$, where A_i is $\mathfrak{p}_i$-secondary. The set $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is independent of the choice of the minimal secondary representation of A. This set is called *the set of attached prime ideals* of A, and denoted by $\mathrm{Att}_R A$. Note also that A has a natural structure as an $\widehat{\mathrm{R}}$-module. With this structure, a subset of A is an R-submodule if and only if it is an $\widehat{\mathrm{R}}$-submodule of A. Therefore, A is an Artinian $\widehat{\mathrm{R}}$-module.

Lemma 2.1. *(i) The set of all minimal elements of* $\mathrm{Att}_R A$ *is exactly the set of all minimal elements of* $\mathrm{Var}(\mathrm{Ann}_R A)$.

(ii) $\mathrm{Att}_R A = \{\widehat{\mathfrak{p}} \cap R \ : \ \widehat{\mathfrak{p}} \in \mathrm{Att}_{\widehat{\mathrm{R}}} A\}$.

R. N. Roberts introduced the concept of Krull dimension for Artinian modules (see [8]). D. Kirby changed the terminology of Roberts and referred to Noetherian dimension to avoid confusion with Krull dimension defined for finitely generated modules (see [9]). The Noetherian dimension of A is denoted by $\text{N-dim}_R(A)$. In this paper, we use the terminology of Kirby (see [9]).

Lemma 2.2 ([1]). *(i)* $\text{N-dim}_R(A) \leqslant \dim(R/\operatorname{Ann}_R A)$, *and the equality holds if A satisfies the property (*).*

(ii) $\text{N-dim}_R(H^i_{\mathfrak{m}}(M)) \le i$, *for all i.*

The following property of attached primes of the local cohomology under localization is known as Weak general Shifted Localization Principle (see [10]).

Lemma 2.3. *We have* $\operatorname{Att}_{R_\mathfrak{p}}(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}))$ *is the subset of* $\{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \min \operatorname{Att}_R(H^i_\mathfrak{m}(M)), \mathfrak{q} \subseteq \mathfrak{p}\}$, *for all* $\mathfrak{p} \in \operatorname{Spec}(R)$.

For an integer $i \ge 0$, following M. Brodmann and R. Y. Sharp (see [11]), the i-th pseudo support of M, denoted by $\operatorname{Psupp}^i_R(M)$, is defined by the set

$$\{\mathfrak{p} \in \operatorname{Spec} R \mid H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}) \neq 0\}.$$

Note that the role of $\operatorname{Psupp}^i_R(M)$ for the Artinian R-module $A = H^i_\mathfrak{m}(M)$ is in some sense similar to that of $\operatorname{Supp} L$ for a finitely generated R-module L, cf. [11], [5]. Although, we always have $\operatorname{Supp} L = \operatorname{Var}(\operatorname{Ann}_R L)$, but the analogous equality $\operatorname{Psupp}^i_R(M) = \operatorname{Var}(\operatorname{Ann}_R H^i_\mathfrak{m}(M))$ is not valid in general. The following lemma gives a necessary and sufficient conditions for the above equality.

Lemma 2.4 ([5]). *Let $i \ge 0$ be an integer. Then the following statements are equivalent:*

(i) $H^i_\mathfrak{m}(M)$ *satisfies the property (*).*

(ii) $\operatorname{Var}\left(\operatorname{Ann}_R(H^i_\mathfrak{m}(M))\right) = \operatorname{Psupp}^i_R M.$

In particular, if $H^i_\mathfrak{m}(M)$ satisfies the property (*) then

$$\min \operatorname{Att}_R(H^i_\mathfrak{m}(M)) = \min \operatorname{Psupp}^i_R M.$$

In 2010, N. T. Cuong, L. T. Nhan and N. T. K. Nga (see [12]) used pseudo support to describe the non-Cohen-Macaulay locus of M. Recall that M is *equidimensional* if $\dim(R/\mathfrak{p}) = d$, for all $\mathfrak{p} \in \min(\operatorname{Ass} M)$.

Lemma 2.5 ([12]). *Suppose that M is equidimensional and the ring $R/\operatorname{Ann}_R M$ is catenary. Then $\operatorname{Psupp}^i_R(M)$ is closed for $i = 0, 1, d$ and* $\operatorname{nCM}(M) = \bigcup_{i=0}^{d-1} \operatorname{Psupp}^i_R(M)$, *where* $\operatorname{nCM}(M)$ *is the Non Cohen-Macaulay locus of M.*

Following M. Nagata ([3]), we say that M is *unmixed* if $\dim(\widehat{R}/\widehat{\mathfrak{p}}) = d$ for all prime ideals $\widehat{\mathfrak{p}} \in \operatorname{Ass} \widehat{M}$, and M is *quasi unmixed* if $\widehat{M}$ is equidimensional. The next lemma show that the property (*) for the local cohomology modules $H^i_\mathfrak{m}(M)$ of levels $i < d$ is closed related to the universal catenaricity and unmixedness of certain local rings.

Lemma 2.6 ([5]). *Assume that $H^i_\mathfrak{m}(M)$ satisfies the property (*) for all $i < d$. Then $R/\mathfrak{p}$ is unmixed for all $\mathfrak{p} \in \operatorname{Ass} M$ and the ring $R/\operatorname{Ann}_R M$ is universally catenary.*

Proof of Theorem 1.2. It follows from the Lemma 2.6 that $R/\mathfrak{p} = R/\operatorname{Ann}_R(R/\mathfrak{p})$ is universally catenary.

Set S to be the image of $R \setminus \mathfrak{p}$ in $\widehat{R}$. We have

$$R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_R \widehat{R} \cong S^{-1}(\widehat{R}/\mathfrak{p}\widehat{R}).$$

We need to prove $(S^{-1}(\widehat{R}/\mathfrak{p}\widehat{R}))_{S^{-1}\widehat{\mathfrak{q}}}$ is Cohen-Macaulay for all $\widehat{\mathfrak{q}} \in \operatorname{Spec}(\widehat{R})$ such that $(\widehat{\mathfrak{q}} \cap R) \cap S = \emptyset$. Assume that the statement is not true. Since

$$(S^{-1}(\widehat{R}/\mathfrak{p}\,\widehat{R}))_{S^{-1}\widehat{\mathfrak{q}}} \cong (\widehat{R}/\mathfrak{p}\,\widehat{R})_{\widehat{\mathfrak{q}}}$$

as $\widehat{R}_{\widehat{\mathfrak{q}}}$-module, there exists $\widehat{\mathfrak{q}} \in \operatorname{Spec}(\widehat{R}), \widehat{\mathfrak{q}} \cap S = \emptyset$ such that $(\widehat{R}/\mathfrak{p}\,\widehat{R})_{\widehat{\mathfrak{q}}}$ is not Cohen-Macaulay. Then there exists $\widehat{\mathfrak{p}} \in \operatorname{Spec}(R), \widehat{\mathfrak{q}} \supseteq \widehat{\mathfrak{p}}$, $(\widehat{\mathfrak{p}} \cap R) \cap S = \emptyset$ and $\widehat{\mathfrak{p}} \in \operatorname{Min}\operatorname{nCM}(\widehat{R}/\widehat{\mathfrak{p}}\widehat{R})$. Hence,

$$\operatorname{nCM}((\widehat{R}/\widehat{\mathfrak{p}}\widehat{R})_{\widehat{\mathfrak{p}}}) = \left\{\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}}\right\}.$$

We have $R/\mathfrak{p}$ is unmixed by Lemma 2.6. So $\widehat{R}/\widehat{\mathfrak{p}}\widehat{R}$ is equidimensional. Hence $(\widehat{R}/\widehat{\mathfrak{p}}\widehat{R})_{\widehat{\mathfrak{p}}}$ is equidimensional. On the other hand, since $(\widehat{R}/\widehat{\mathfrak{p}}\widehat{R})_{\widehat{\mathfrak{p}}}$ is the image of a Cohen-Macaulay ring, $(\widehat{R}/\widehat{\mathfrak{p}}\widehat{R})_{\widehat{\mathfrak{p}}}$ is generalized Cohen-Macaulay.

Set $s = \dim \widehat{R}/\widehat{\mathfrak{p}}\widehat{R} = \operatorname{ht}(\widehat{\mathfrak{p}}/\mathfrak{p}\,\widehat{R})$. By Lemma 2.5, we have

$$\operatorname{nCM}(\widehat{R}/\widehat{\mathfrak{p}}\widehat{R})_{\widehat{\mathfrak{p}}} = \bigcup_{i=0}^{s-1} \operatorname{Psupp}^i_{\widehat{R}}((\widehat{R}/\widehat{\mathfrak{p}}\widehat{R})_{\widehat{\mathfrak{p}}}).$$

Therefore, there exists $i < s$ such that $H^i_{\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}}}(\widehat{R}/\mathfrak{p}\,\widehat{R})_{\widehat{p}} \neq 0$. On the other hand,

$$\ell(H^i_{\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}}}(\widehat{R}/\mathfrak{p}\,\widehat{R})_{\widehat{p}}) < \infty.$$

Then

$$\operatorname{Att}_{\widehat{R}}(H^i_{\widehat{\mathfrak{p}}\widehat{R}_{\widehat{\mathfrak{p}}}}(\widehat{R}/\mathfrak{p}\,\widehat{R})_{\widehat{p}}) = \left\{\mathfrak{p}\,\widehat{R}_{\widehat{\mathfrak{p}}}\right\}.$$

It is followed by Weak general Shifted Localization Principle (Lemma 2.3) that $\widehat{\mathfrak{p}} \in \operatorname{Att}_{\widehat{R}}(H^{i+\dim \widehat{R}/\widehat{\mathfrak{p}}}_{\mathfrak{m}}(\widehat{R}/\mathfrak{p}\,\widehat{R}))$. Set $j = i + \dim \widehat{R}/\widehat{\mathfrak{p}}$. We have

$$\begin{aligned} j < \operatorname{ht}\widehat{\mathfrak{p}}/\mathfrak{p}\,\widehat{R} + \dim \widehat{R}/\widehat{\mathfrak{p}} &\leq \dim \widehat{R}/\mathfrak{p}\,\widehat{R} \\ &= \dim R/\mathfrak{p}\,. \end{aligned}$$

Hence, $\mathfrak{p} \in \operatorname{Att}_R(H^j_{\mathfrak{m}}(R/\mathfrak{p}))$ by Lemma 2.1. By Lemma 2.2

$$\begin{aligned} \text{N-dim}\, H^j_{\mathfrak{m}}(R/\mathfrak{p}) \leq j &< \dim R/\mathfrak{p} \\ &\leq R/\operatorname{Ann}_R H^j_{\mathfrak{m}}(R/\mathfrak{p}). \end{aligned}$$

This implies that $H^j_{\mathfrak{m}}(R/\mathfrak{p})$ does not satisfy the property (*). It is in contradiction to the hypothesis. Therefore, all its formal fibers over $\mathfrak{p}$ are Cohen-Macaulay. □

3. Conclusion

The paper gives a relation between the property (*) of local cohomology module and structure of base ring. In detail, we prove that for each $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $H^i_{\mathfrak{m}}(R/\mathfrak{p})$ satisfies the property (*) for all i, then $R/\mathfrak{p}$ is universally catenary and the formal fibre of R over $\mathfrak{p}$ is Cohen-Macaulay.

References

[1]. C. T. Nguyen and N. T. Le, "On the Noetherian dimension of Artinian modules," *Vietnam Journal of Mathematics*, vol. 30, no. 2, pp. 121-130, 2002.

[2]. D. Ferrand and M. Raynaud, "Fibres formelles d'un anneau local Noetherian," *Annales Scientifiques de l'École Normale Supérieure*, vol. 3, no. 4, pp. 295-311, 1970.

[3]. M. Nagata, *Local rings*, Interscience, New York, 1962.

[4]. C. T. Nguyen, D. T. Nguyen and N. T. Le, "Top local cohomology and the catenaricity of the unmixed support of a finitely generated module," *Communications in Algebra*, vol. 35, no. 5, pp. 1691-1701, 2007.

[5]. N. T. Le and A. N. Tran, "On the unmixedness and the universal catenaricity of local rings and local cohomology modules,"*Journal of Algebra*, vol. 321, pp. 303-311, 2009 .

[6]. N. T. Le and C. D. M. Tran, "Noetherian dimension and co-localization of Artinian modules over local rings," *Algebra Colloquium*, vol. 21, pp. 663-670, 2014.

[7]. I. G. Macdonald, "Secondary representation of modules over a commutative ring," *Symposia Mathematica*, vol. 11 , pp. 23-43, 1973.

[8]. R. N. Roberts, "Krull dimension for Artinian modules over quasi local commutative rings," *Quarterly Journal of Mathematics*, vol. 26, no. 2, pp. 269-273, 1975.

[9]. D. Kirby, "Dimension and length of Artinian modules," *Quarterly Journal of Mathematics*, vol. 41, no. 2, pp. 419-429, 1990.

[10]. M. Brodmann and R. Y. Sharp, *Local cohomology: an algebraic introduction with geometric applications*, Cambridge University Press, 1998.

[11]. M. Brodmann and R. Y. Sharp, "On the dimension and multiplicity of local cohomology modules," *Nagoya Mathematical Journal*, vol. 167, pp. 217-233, 2002.

[12]. C. T. Nguyen, N. T. Le and N. K. T. Nguyen, "On pseudo supports and non-Cohen-Macaulay locus of finitely generated modules," *Journal of Algebra*, vol. 323, pp. 3029-3038, 2010.