ON INTERMEDIATE RINGS WHICH ARE FINITELY GENERATED MODULES OVER A NOETHERIAN RING

Nguyen Xuan Linh[1], Luu Phuong Thao[2*]
[1]*National University of Civil Engineering*
[2]*TNU - University of Education*

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ABSTRACT

Let (R, m) be a commutative Noetherian ring and $Q(R)$ the total quotient ring of R. The aim of this paper is to study the structure of intermediate rings between R and $Q(R)$. Let X be the set of all equivalent classes $[I]$, where I is an ideal of R such that I 2 = aI for some non zero divisor $a \in I$. Let Y be the set of all intermediate rings A between R and $Q(R)$ such that A is finitely generated R-modules. In this paper, we establish a bijection from X to Y. Some examples are given to clarify the result. Firstly, we show that if R is a principal ideal domain, then R is the unique element of Y. Secondly, we give a Buchsbaum ring R which is not Cohen-Macaulay and we construct a Cohen-Macaulay intermediate ring $A \in Y$. In order to solve the problem, we apply the method investigated by S. Goto in 1983, L. T. Nhan and M. Brodmann 2012.

VỀ VÀNH TRUNG GIAN LÀ MÔĐUN HỮU HẠN SINH TRÊN VÀNH NOETHER

Nguyễn Xuân Linh[1], Lưu Phương Thảo[2*]
[1]*Trường Đại học Xây dựng*
[2]*Trường Đại học Sư phạm - ĐH Thái Nguyên*

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TỪ KHÓA

Vành các thương toàn phần
Vành trung gian
Vành Noether
Môđun Cohen-Macaulay
Môđun hữu hạn sinh

TÓM TẮT

Cho (R, m) là vành giao hoán Noether và $Q(R)$ là vành các thương toàn phần của R. Mục đích của bài báo này là nghiên cứu cấu trúc của các vành trung gian giữa R và $Q(R)$. Gọi X là tập tất cả các lớp tương đương $[I]$, trong đó I là ideal của R sao cho I 2 = aI với $a \in I$ là phần tử không là ước của không trong R. Gọi Y là tập tất cả các vành trung gian A giữa R và $Q(R)$ sao cho A là R-môđun hữu hạn sinh. Trong bài báo này, chúng tôi thiết lập một song ánh từ X đến Y. Một số ví dụ được đưa ra để làm rõ kết quả. Thứ nhất, chúng tôi chỉ ra nếu R là một miền ideal chính thì R là phần tử duy nhất của Y. Thứ hai, cho một vành Buchsbaum R mà không là Cohen-Macaulay, chúng tôi xây dựng một vành trung gian Cohen-Macaulay $A \in Y$. Để giải quyết vấn đề, chúng tôi áp dụng phương pháp nghiên cứu của S. Goto năm 1983, L. T. Nhàn và M. Brodmann 2012.

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* Corresponding author. *Email: thaolp@tnue.edu.vn*

1. Introduction

Throughout this paper, let R be a commutative Noetherian ring. Denoted by $Q(R)$ the total quotient ring of R, i.e. $Q(R) = S^{-1}R$ is the ring of fractions of R with respect to the multiplicative closed subset S, where S is the set of all non zero divisors of R.

The study of intermediate rings between R and $Q(R)$ which are finitely generated R-module gives a lot of information on R ([1], [2]). In this paper, we discribe the structure of these intermediate rings. Let $\mathcal{X}$ be the set of all equivalent classes $[I]$, where I is an ideal of R such that $I^2 = aI$ for some non zero divisor $a \in I$ (see Lemma 2.3). Let $\mathcal{Y}$ be the set of all intermediate rings between R and $Q(R)$ which are finitely generated R-modules.

The following theorem is the main result of this paper.

Main Theorem. *There is a bijection φ from $\mathcal{X}$ to $\mathcal{Y}$ given by $\varphi([I]) = \dfrac{I}{a}$, where $a \in I$ is a non zero divisor in R satisfying $I^2 = aI$.*

We also present some examples to clarify the result. Firstly, we prove that if R is a principal ideal domain, then R is the unique element of $\mathcal{Y}$. Secondly, by using the method of S. Goto [1] and An-Nhan-Thao [3], we give a Buchsbaum ring R which is not Cohen-Macaulay and we construct a Cohen-Macaulay intermediate ring $A \in \mathcal{Y}$.

2. Proof of the main result

For each intermediate ring A between R and $Q(R)$, it is clear that A has a natural structure as an R-module induced by the natural injection from R to A.

Lemma 2.1. *Let I be an ideal of R. Suppose that there exists $a \in I$ such that a is not a zero divisor in R and $aI = I^2$. Set*

$$\frac{I}{a} = \left\{ \frac{x}{a} \in Q(R) \mid x \in I \right\}.$$

Then $\dfrac{I}{a}$ is an intermidiate ring between R and $Q(R)$ which is a finitely generated R-module.

Proof. In order to prove $\dfrac{I}{a}$ is a subring of $Q(R)$, we prove that $\dfrac{I}{a}$ is closed under the addtion, the multiplication and $\dfrac{-1}{1} \in \dfrac{I}{a}$. Let $\dfrac{b}{a}, \dfrac{c}{a} \in \dfrac{I}{a}$, where $b, c \in I$. Since $b + c \in I$, we have

$$\frac{b}{a} + \frac{c}{a} = \frac{b+c}{a} \in \frac{I}{a}.$$

Because $aI = I^2$, there exists $d \in I$ such that $bc = ad$. Therefore,

$$\frac{b}{a}.\frac{c}{a} = \frac{bc}{a^2} = \frac{ad}{a^2} = \frac{d}{a} \in \frac{I}{a}.$$

It is clear that $\dfrac{-1}{1} = \dfrac{-a}{a} \in \dfrac{I}{a}$. Hence $\dfrac{I}{a}$ is a subring of $Q(R)$. Let $x \in R$. Then

$$x = \frac{x}{1} = \frac{ax}{a} \in \frac{I}{a}.$$

Thus, $\dfrac{I}{a}$ is a intermediate ring between R and $Q(R)$. Since R is a Noetherian ring, I is finitely generated. Let $a_1, a_2, \ldots, a_n \in I$ such that

$$I = \big(a_1, a_2, \ldots, a_n\big)R.$$

It is clear that $\dfrac{I}{a} = \left(\dfrac{a_1}{a}, \dfrac{a_2}{a}, \ldots, \dfrac{a_n}{a}\right)R$. Therefore, $\dfrac{I}{a}$ is a finitely generated R-module. $\square$

It should be mentioned that, with the assumtion and notation as in Lemma 2.1, if $(R, \mathfrak{m})$ is a Noetherian local ring and $a \in R \setminus \mathfrak{m}$ then a is a unit. Therefore, $I = R$. Since a is a unit, we have $\dfrac{R}{a} = R$. Thus, $\dfrac{I}{a} = \dfrac{R}{a} = R$.

From now on, we denote by $\mathcal{I}$ the set of all ideal I of R such that $I^2 = aI$ for some non zero divisor $a \in I$. Below we show that the intermediate ring $\dfrac{I}{a}$ defined in Lemma 2.1 does not depend on the choice of the non zero divisor $a \in I$.

Lemma 2.2. *Let ideal $I \in \mathcal{I}$ and $a, b \in I$ be non zero divisors such that $I^2 = aI$, $I^2 = bI$. Then, in the total quotient ring $Q(R)$, the two intermediate rings $\dfrac{I}{a}$ and $\dfrac{I}{b}$ are the same.*

Proof. Let $\dfrac{x}{a} \in \dfrac{I}{a}$, where $x \in I$. We have $I^2 = aI$, there exists $y \in I$ such that $xb = ay$. Hence

$$\frac{x}{a} = \frac{xb}{ab} = \frac{ay}{ab} = \frac{y}{b} \in \frac{I}{b}.$$

Therefore, $\dfrac{I}{a} \subseteq \dfrac{I}{b}$. By the same arguments, we get the converse inclusion. $\square$

Lemma 2.3. *Let $I, J \in \mathcal{I}, a \in I, b \in J$ be non zero divisors such that $I^2 = aI, J^2 = bJ$. We define the relation on $\mathcal{I}$ as follows: I is related to J if and only if the two intermediate rings $\dfrac{I}{a}$ and $\dfrac{J}{b}$ between R and $Q(R)$ are the same. Then this relation is an equivalent relation on $\mathcal{I}$.*

Proof. Let $I, J, K \in \mathcal{I}$ and three elements $a \in I, b \in J, c \in K$ be non zero divisors such that

$$I^2 = aI, J^2 = bI, K^2 = cK.$$

It is clear that $\dfrac{I}{a} = \dfrac{I}{a}$. So, I is related to I. Suppose that I is related to J. Then $\dfrac{I}{a} = \dfrac{J}{b}$. Hence $\dfrac{J}{b} = \dfrac{I}{a}$. Therefore, J is related to I. Suppose that I is related to J, and J is related to K. Then $\dfrac{I}{a} = \dfrac{J}{b}$ and $\dfrac{J}{b} = \dfrac{K}{c}$. Therefore, $\dfrac{I}{a} = \dfrac{K}{c}$, i.e. I is related to K. $\square$

From now on, for each ideal $I \in \mathcal{I}$, we denote by $[I]$ is the equivalent class of I with respect to the equivalent relation defined in Lemma 2.3. Set $\mathcal{X} = \{[I] \mid I \in \mathcal{I}\}$. Let $\mathcal{Y}$ be the set of all intermediate rings between R and $Q(R)$ which are finitely generated R-modules. The following theorem is the main result in this paper.

Theorem 2.4. *There is a bijection φ from $\mathcal{X}$ to $\mathcal{Y}$ given by $\varphi([I]) = \dfrac{I}{a}$, where $a \in I$ is a non zero divisor in R satisfying $I^2 = aI$.*

Proof. It follows by Lemma 2.2 that φ is a map from $\mathcal{X}$ to $\mathcal{Y}$. Morever, the map φ is an injection by Lemma 2.3. Now we prove that φ is a surjection. Let $A \in \mathcal{Y}$. Since A is a subring of $Q(R)$ and A is a finitely generated R-module, we can write

$$A = \left(\frac{x_1}{a_1}, \frac{x_2}{a_2}, \dots, \frac{x_n}{a_n}\right)R,$$

where $x_i \in R$ and all $a_i \in R$ are non zero divisors for all $i = 1, 2, \dots, n$. Put $a = a_1 \cdots a_n$. Then a is also a non zero divisor in R. Set

$$I = \left\{x \in R \mid \frac{x}{a} \in A\right\}.$$

We prove that I is an ideal of R. Since A is a subring of $Q(R)$, we get $\dfrac{1}{1} \in A$. Hence $\dfrac{a}{a} \in A$. It follows that $a \in I$. Let $x, y \in I$. Then $\dfrac{x}{a}, \dfrac{y}{a} \in A$. Since A is a subring of $Q(R)$, we have $\dfrac{x}{a} - \dfrac{y}{a} = \dfrac{x-y}{a} \in A$. Hence $x - y \in I$. Let $x \in I$ and $r \in R$. Then $\dfrac{x}{a} \in A$. Note that A is an intermediate ring between R and $Q(R)$, we have $r = \dfrac{r}{1} \in A$ and hence $\dfrac{r}{1}.\dfrac{x}{a} = \dfrac{rx}{a} \in A$. So, $rx \in I$. Therefore, I is an ideal of R.

We prove $I^2 = aI$. Since $a \in I$, we have $aI \subseteq I^2$. Let $x, y \in I$. Then $\dfrac{x}{a}, \dfrac{y}{a} \in A$. Hence $\dfrac{xy}{a^2} \in A$ as A is a subring of $Q(R)$. Since $A = \left(\dfrac{x_1}{a_1}, \dfrac{x_2}{a_2}, \dots, \dfrac{x_n}{a_n}\right)R$, there exist $c_1, \dots, c_n \in R$ such that

$$\frac{xy}{a^2} = \sum_{i=1}^{n} \frac{x_i}{a_i} c_i.$$

Set

$$z = a \sum_{i=1}^{n} \frac{x_i}{a_i} c_i.$$

As $a = a_1 \dots a_n$, it follows that $z \in R$. Hence, $\dfrac{z}{a} = \dfrac{xy}{a^2} \in A$. Hence $z \in I$. Hence $xy = az \in aI$. Therefore, $I^2 \subseteq aI$. Thus $I^2 = aI$.

Finally, we prove $\varphi([I]) = A$. We have $\varphi([I]) = \dfrac{I}{a} \subseteq A$ by the definition of I. Suppose that $\dfrac{c}{u} \in A$, where $c, u \in R$ and u is a non zero divisor. Then there exist $d_i \in R$, $i = 1, \dots, n$, such that

$$\frac{c}{u} = \sum_{i=1}^{n} \frac{x_i}{a_i} d_i.$$

As $a = a_1 \dots a_n$, we can write

$$\frac{c}{u} = \sum_{i=1}^{n} \frac{x_i}{a_i} d_i = \frac{x}{a}$$

for some $x \in R$. Since $\frac{c}{u} \in A$, we have $\frac{x}{a} \in A$. Hence $x \in I$ and hence $\frac{c}{u} = \frac{x}{a} \in \frac{I}{a}$. Therefore, $A \subseteq \frac{I}{a}$. Thus, $A = \frac{I}{a}$. □

Now, we give examples to clarify the result.

Example 2.5. *Let R be a principal ideal domain (for example $R = \mathbb{Z}$; $R = k[x]$; or $R = k[[x]]$ for some field k). Then R is the unique intermediate ring between R and $Q(R)$ which is a finitely generated module over R.*

Proof. Let I be an ideal of R such that $I^2 = aI$ for some non zero divisor $a \in I$. Since every ideal of R is principal, there exists $b \in I$ such that $I = Rb$. Hence $Rb^2 = Rab$. Note that $a \neq 0$. Hence $I = Rb \neq 0$ and hence $b \neq 0$. As $a \in I = Rb$, we have that a is a multiple of b. Since $Rb^2 = Rab$, we have $b^2 = uab$ for some $u \in R$. Since $b \neq 0$ and R is a domain, we have $b = ua$, it means that b is multiple of a. Hence u is a unit. So

$$\frac{I}{a} = \frac{Rua}{a} = Ru = R.$$

Thus, the result follows by Theorem 2.4. □

From now on, let $(R, \mathfrak{m})$ is a Noetherian local ring. We say that M is a Cohen-Macaulay module if $\dim_R(M) = \operatorname{depth}_R(M)$. Note that Cohen - Macaulay modules play an important role in many different fields of mathematics. The studying of intermediate rings A between R and $Q(R)$ such that A is a Cohen-Macaulay ring and A is a finitely generated R-module has been interested by many mathematicians ([1], [2], [3]). The following example discribes such an intermediate Cohen- Macaulay ring.

Example 2.6. *Let $S = k[[x, y, z, t, u, v]]$ be the formal power series ring of 6 variables over a field k. Let $R = \frac{S}{(x, y, z) \cap (t, u, v)}$. For $f \in S$, denote by $\overline{f}$ is the image of f in R. Let $a = \overline{x + t}$, $\mathfrak{m} = (x, y, z, t, u, v)R$, $I = \bigcup_{n \geq 0} (aR :_R \mathfrak{m}^n)$. Set $A = \frac{I}{a}$. Then*

(i) R is local with the maximal ideal $\mathfrak{m}$.

(ii) $a \in I$ is a non zero divisor in R and $aI = I^2$.

(iii) A is an intermediate Cohen-Macaulay ring between R and $Q(R)$, and A is a finitely generated R-module.

Proof. From the exact sequence

$$0 \to R \to \frac{S}{(x, y, z)S} \oplus \frac{S}{(t, u, v)S} \to \frac{S}{(x, y, z, t, u, v)S} \to 0$$

with notice that $\dim_R \dfrac{S}{(x,y,z)S} = 3, \operatorname{depth}_R \dfrac{S}{(x,y,z)S} = 3, \dim_R \dfrac{S}{(t,u,v)S} = 3, \operatorname{depth}_R \dfrac{S}{(t,u,v)S} =$ 3, we have $H^0_m(R) = 0, H^2_m(R) = 0$ and $H^1_m(R) \cong \dfrac{S}{(x,y,z,t,u,v)S}$. Set

$$I(R) = \sup_{\underline{x}} \Big(l\big(R/\underline{x}R\big) - e(\underline{x}, R) \Big),$$

where $\underline{x}$ runs over all systems of parameters of R. Then we get by [5] that

$$I(R) = \sum_{i=0}^{2} \binom{2}{i-1} l(H^i_m(R)) = 1.$$

Therefore, R is Buchsbaum (see [1]). We have $\operatorname{Ann}_R H^1_m(R) = (x,y,z,t,u,v)R = \mathfrak{m}$. Note that $\operatorname{Ass}(R) = \{(x,y,z)R, (t,u,v)R\}$. Hence $a \notin (x,y,z)R \cup (t,u,v)R = \operatorname{NZD}(R)$. Choose $I = \bigcup\limits_{n \geq 0} (aR :_R \mathfrak{m}^n)$. Then I is an ideal of R and $a \in I$. We have $aI = I^2$ by [1]. Thus $A = \dfrac{I}{a}$ is an intermediate Cohen-Macaulay ring between R and $Q(R)$ and A is a finitely generated R-module (see [1]). $\square$

3. Conclusion

In this paper, we prove that there is a bijection from $\mathcal{X}$ to $\mathcal{Y}$, where $\mathcal{X}$ is the set of all equivalent classes $[I]$ with an ideal I of R such that $I^2 = aI$ for some $a \in \operatorname{NZD}(R)$ and $\mathcal{Y}$ is the set of all intermediate rings A between R and $Q(R)$ such that A is a finitely generated R-module. Some examples are presented to make clear the result.

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