

AN ITERATIVE ALGORITHM FOR FINDING THE MINIMUM NORM POINT IN THE SOLUTION SET OF SPLIT FIXED POINT PROBLEM

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ABSTRACT

The split feasibility problem and the variational inequality problem have many practical applications in signal processing, image reconstruction intensity-modulated radiation therapy, optimal control theory and many other fields. The problem we consider here is a bilevel problem, when the leader problem is the minimum norm point problem and the follower one is the split fixed point problem. The minimum norm point problem is a particular case of the variational inequality problem, when the cost mapping is the unit operator of the Hilbert space. In this paper, we propose an iterative method for approximating the solution of the bilevel problem. This method bases on the result presented by Tran Viet Anh and Le Dung Muu in 2016, which is a combination between the projection method for variational inequality and the Krasnoselskii–Mann scheme for fixed points of nonexpansive mappings. The strong convergence of the method is proven. We close the paper by considering an example to illustrate the strong convergence of method.

MỘT PHƯƠNG PHÁP LẬP HIỆN TÌM ĐIỂM CÓ CHUẨN NHỎ NHẤT TRÊN TẬP NGHIỆM CỦA BÀI TOÁN ĐIỂM BẤT ĐỘNG TÁCH

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TỪ KHÓA

Bài toán chấp nhận tách
Bất đẳng thức biến phân
Không gian Hilbert
Ánh xạ không giãn
Điểm bất động

TÓM TẮT

Bài toán chấp nhận tách và bài toán bất đẳng thức biến phân có nhiều ứng dụng quan trọng trong các lĩnh vực như xử lý tín hiệu, xử lý ảnh, điều khiển tối ưu và nhiều lĩnh vực khác. Ở đây, chúng tôi quan tâm tới một bài toán hai cấp, đó là bài toán tìm điểm có chuẩn nhỏ nhất trên tập nghiệm của bài toán điểm bất động tách. Bài toán tìm điểm có chuẩn nhỏ nhất là trường hợp riêng của bài toán bất đẳng thức biến phân, trong đó ánh xạ giá F là ánh xạ đồng nhất của không gian Hilbert. Trong bài báo này, chúng tôi trình bày một phương pháp lập hiện để xấp xỉ nghiệm của bài toán trên. Phương pháp được xây dựng dựa trên kết quả đã được trình bày bởi các tác giả Trần Việt Anh và Lê Dũng Mưu năm 2016, đó là sự kết hợp giữa phương pháp chiếu giải bài toán bất đẳng thức biến phân và phương pháp Krasnoselskii–Mann giải bài toán điểm bất động của các ánh xạ không giãn. Định lý về sự hội tụ mạnh của thuật toán được chứng minh. Ở cuối bài báo, chúng tôi trình bày một ví dụ số để minh họa cho sự hội tụ của phương pháp.

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1 Introduction

Throughout the paper, we use $\mathcal{H}$ to denote the real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let C and Q are nonempty, closed and convex subsets of the Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a bounded linear operator. The *split feasibility problem* is defined as follow:

$$\text{Find } x^* \in C \text{ such that } Ax^* \in Q. \quad (1.1)$$

This problem was first introduced by Censor and Elfving [1] in 1994. Thenceforth, there are many split problems when C and Q are some specific sets, such as the solution set of variational inequality, the set of fixed points of nonexpansive mapping. . . In 2002, Byrne [2] advanced an iterative method to approximate the solution of (1.1), which is often called *CQ algorithm*. The algorithm is defined as:

$$x^{k+1} = P_C(x^k + \gamma A^\top (P_Q - I)Ax^k), \quad k \geq 0, \quad (1.2)$$

where C and Q are nonempty, closed and convex sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, A is a real $m \times n$ matrix, $A^\top$ is the transpose of A , $\gamma \in \left(0, \frac{2}{L}\right)$, where L denotes the largest eigenvalue of the matrix $A^\top A$. This algorithm thus can be implemented if the projection onto C and Q can be computed with a reasonable effort. In 2010, Xu [3] considered the split feasibility problem setting in infinite-dimensional Hilbert space, then the CQ algorithm became:

$$x^{k+1} = P_C(x^k + \gamma A^*(P_Q - I)Ax^k), \quad k \geq 0,$$

where $\gamma \in \left(0, \frac{2}{\|A\|^2}\right)$ and A^* is the adjoint operator of A .

Now consider the variational inequality problem in Hilbert space. The theory of variational inequality problem (in short VIP) was first considered by Stampacchia [4, 5] in the early 1970s. Since then, it has played the important rule in optimization and nonlinear analysis. To be practice, let C be a nonempty, closed and convex subset of the Hilbert space $\mathcal{H}$, and $F : \mathcal{H} \rightarrow \mathcal{H}$ be a mapping. Then the VIP is defined as follow:

$$\text{Find } x^* \in C \text{ such that } \langle F(x^*), x - x^* \rangle \geq 0 \quad \forall x \in C. \quad (\text{VIP})$$

The VIP defined for C and F can be denoted by $\text{VIP}(C, F)$. When $F = I^\mathcal{H}$ (the unit operator in $\mathcal{H}$), by the Cauchy–Schwarz inequality, the $\text{VIP}(C, I^\mathcal{H})$ becomes the minimum norm point problem (in short, MNPP), denoted by $\text{MNPP}(C)$, which takes the form: $\text{find } x^* = \arg \min_C \|x\|$.

In this paper, we consider a bilevel problem, where the leader is the minimum norm point problem and the follower one is the split fixed point problem. To be practice, let C and Q are two nonempty closed convex sets in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, $T : C \rightarrow C$ and $S : Q \rightarrow Q$ are nonexpansive mappings, and $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a bounded linear operator. Then the problem can be formulated as follow:

$$\text{Find } x^* = \arg \min_{\Omega} \|x\|, \quad (\text{MNPP})$$

where Ω is the solution set of the split fixed point problem: $\text{Find } x^* \in C$ such that

$$x^* = Tx^*, Ax^* \in Q, \text{ and } Ax^* = S(Ax^*). \quad (\text{SFP})$$

Generally, the solution of this problem is the projection of the origin of $\mathcal{H}_1$ onto Ω . Because of nonexpansiveness of T and S , Ω is nonempty, closed and convex, then the solution exists uniquely. Denoting the solution set of $\text{MNPP}(\Omega)$ by $\text{Sol}(\Omega)$. However, the projection onto Ω cannot be computed. In this paper, based on the method of Tran Viet Anh and Le Dung Muu [6], we propose an algorithm to solve the (MNPP)–(SFP), which can be considered as a combination of projection method for variational inequality and the Krasnoselskii–Mann method [7] for fixed point problem.

The remaining part of this paper is organized as follows: the next section are some notations, definitions and lemmas that will be used for the validity and convergence of the algorithm. The third section is devoted to the description of our proposed algorithm and its strong convergence result. In Section 4, we applied our results to study the minimum norm point in solution set of split variational inequality problem. Finally, we illustrate the proposed method by considering two numerical experiments.

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space. In what follows, we write $x^k \rightarrow x$ to indicate that the sequence $\{x^k\}$ converges strongly to x while $x^k \rightharpoonup x$ to indicate that the sequence $\{x^k\}$ converges weakly to x .

Definition 2.1 ([8]). A mapping $F : C \rightarrow C$ is said to be L -Lipschitz continuous if there exists a positive constant L such that $\|F(x) - F(y)\| \leq L\|x - y\|$, $\forall x, y \in C$. If $0 < L < 1$, F is said to be contraction mapping. If $L = 1$, F is said to be nonexpansive mapping.

Definition 2.2 ([9]). Let $F : C \rightarrow C$ be a nonexpansive mapping. A point $x \in C$ is said to be a fixed point of F if $F(x) = x$.

Throughout this paper, we denote the set of all fixed points of F by $\text{Fix}(F)$, i.e. $\text{Fix}(F) = \{x \in C \mid F(x) = x\}$.

Definition 2.3 ([9]). Let C be a set in the Hilbert space $\mathcal{H}$. For each $x \in \mathcal{H}$, $P_C(x)$ is an element in C such that: $\|x - P_C(x)\| \leq \|x - y\|$, $\forall y \in C$. The mapping $P_C : \mathcal{H} \rightarrow C$ is called the metric projection from $\mathcal{H}$ onto C . In practice, we usually calculate the metric projection of a given point onto a nonempty closed convex set C , then we have the following theorem:

Theorem 2.1 (Projection Theorem). *Let C be a nonempty closed convex subset of $\mathcal{H}$, and let z be a point in $\mathcal{H}$. There exists a unique point that minimizes $\|z - x\|$ over $x \in C$, it is the metric projection of z onto C .*

When Ω is nonempty closed convex, and $z = 0$, we get the uniqueness of the solution of $\text{MNPP}(\Omega)$. The metric projection has the following property: $\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y \rangle$, $\forall x, y \in \mathcal{H}$. Then P_C is nonexpansive.

The following lemmas will be used to prove the strong convergence of the proposed method.

Lemma 2.1 ([10]). *Let $\mathcal{H}$ be a real Hilbert space. For all $x, y \in \mathcal{H}$, we have*

$$2\langle x, y \rangle = \|x\|^2 + \|y\|^2 - \|x - y\|^2 = \|x + y\|^2 - \|x\|^2 - \|y\|^2.$$

Lemma 2.2 ([11]). *Let $\{x^n\}$ and $\{y^n\}$ be bounded sequences in a Banach space X and $\{\alpha_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \rightarrow \infty} \alpha_n \leq \limsup_{n \rightarrow \infty} \alpha_n < 1$. Suppose that $x^{n+1} = \alpha_n x^n + (1 - \alpha_n)y^n$, $\forall n \geq 0$ and $\limsup_{n \rightarrow \infty} (\|y^{n+1} - y^n\| - \|x^{n+1} - x^n\|) \leq 0$. Then $\lim_{n \rightarrow \infty} \|y^n - x^n\| = 0$.*

Lemma 2.3 (Opial's lemma, [6]). *For any sequence $\{x^n\} \subset \mathcal{H}$ with $x^n \rightharpoonup x$, the inequality*

$$\liminf_{n \rightarrow \infty} \|x^n - x\| < \liminf_{n \rightarrow \infty} \|x^n - y\|$$

holds for any $y \in \mathcal{H} \setminus \{x\}$.

Lemma 2.4 ([12]). *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\theta_n, \quad \forall n \geq 0,$$

where $\{\gamma_n\}$, $\{\theta_n\}$ are sequences of real numbers such that (a) $\{\gamma_n\} \subset (0, 1)$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$, (b) $\limsup_{n \rightarrow \infty} \theta_n \leq 0$. Then $\lim_{n \rightarrow \infty} a_n = 0$.

3 Main Results

In this section, we present an iterative algorithm to approximate the solution of (MNPP)–(SFP) and prove its strong convergence.

Theorem 3.1. *Let C and Q be two nonempty closed convex subset of two real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and let $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be a bounded linear operator with its adjoint A^* . Let $T : C \rightarrow C$ and $S : Q \rightarrow Q$ be nonexpansive mappings. For a given $x^0 \in C$, let the iterative sequences $\{x^k\}$, $\{u^k\}$, $\{y^k\}$ and $\{z^k\}$ be generated by*

$$\begin{cases} u^k = P_Q(Ax^k), \\ y^k = P_C(x^k + \delta A^*(Su^k - Ax^k)), \\ z^k = P_C((1 - \lambda_k \mu)y^k), \\ x^{k+1} = \alpha_k x^k + (1 - \alpha_k)T(z^k) \quad \forall k \geq 0, \end{cases} \quad (3.1)$$

where $\delta \in \left(0, \frac{1}{\|A\|^2 + 1}\right)$, $\mu \in \left(0, \frac{2\beta}{L^2}\right)$, $\{\lambda_k\}$ and $\{\alpha_k\}$ are sequences in $(0, 1)$ satisfying the following control conditions: (a) $\lim_{k \rightarrow \infty} \lambda_k = 0$, (b) $\sum_{k=0}^{\infty} \lambda_k(1 - \alpha_k) = \infty$, (c) $\lim_{k \rightarrow \infty} \alpha_k = \alpha \in (0, 1)$. Suppose that $\Omega \neq \emptyset$ then the sequence $\{x^k\}$ converges strongly to the unique solution x^* of the bilevel problem (MNPP)–(SFP).

Proof. Let $x^* \in \text{Sol}(\Omega)$, we have $x^* \in \Omega$, i.e. $x^* \in C$, $Ax^* \in Q$, $Tx^* = x^*$ and $S(Ax^*) = Ax^*$.

By the firmly nonexpansiveness of P_Q , we obtain

$$\begin{aligned} \|u^k - Ax^*\|^2 &= \|P_Q(Ax^k) - P_Q(Ax^*)\|^2 \leq \langle P_Q(Ax^k) - P_Q(Ax^*), Ax^k - Ax^* \rangle \\ &= \langle u^k - Ax^*, Ax^k - Ax^* \rangle = \frac{1}{2} [\|u^k - Ax^*\|^2 + \|Ax^k - Ax^*\|^2 - \|u^k - Ax^k\|^2]. \end{aligned}$$

Consequently, we get

$$\|u^k - Ax^*\|^2 \leq \|Ax^k - Ax^*\|^2 - \|u^k - Ax^k\|^2. \quad (3.2)$$

Thanks to the nonexpansiveness of S , $S(Ax^*) = Ax^*$ and (3.2), we can write

$$\|Su^k - Ax^*\|^2 = \|Su^k - S(Ax^*)\|^2 \leq \|u^k - Ax^*\|^2 \leq \|Ax^k - Ax^*\|^2 - \|u^k - Ax^k\|^2.$$

Therefore,

$$\|Su^k - Ax^*\|^2 - \|Ax^k - Ax^*\|^2 \leq -\|u^k - Ax^k\|^2. \quad (3.3)$$

It follows from (3.3) that

$$\begin{aligned} \langle A(x^k - x^*), Su^k - Ax^k \rangle &= \langle A(x^k - x^*) + Su^k - Ax^k - (Su^k - Ax^k), Su^k - Ax^k \rangle \\ &= \langle Su^k - Ax^*, Su^k - Ax^k \rangle - \|Su^k - Ax^k\|^2 \\ &= \frac{1}{2} [\|Su^k - Ax^*\|^2 + \|Su^k - Ax^k\|^2 - \|Ax^k - Ax^*\|^2] - \|Su^k - Ax^k\|^2 \\ &= \frac{1}{2} [\|Su^k - Ax^*\|^2 - \|Ax^k - Ax^*\|^2] - \|Su^k - Ax^k\|^2 \\ &\leq -\frac{1}{2} \|u^k - Ax^k\|^2 - \frac{1}{2} \|Su^k - Ax^k\|^2. \end{aligned}$$

Then

$$2\delta \langle A(x^k - x^*), Su^k - Ax^k \rangle \leq -\delta \|u^k - Ax^k\|^2 - \delta \|Su^k - Ax^k\|^2.$$

Using the above inequality and the fact that the projection P_C is nonexpansive, we obtain

$$\|y^k - x^*\|^2 = \|P_C(x^k + \delta A^*(Su^k - Ax^k)) - P_C(x^*)\|^2 \quad (3.4)$$

$$\begin{aligned} &\leq \|(x^k - x^*) + \delta A^*(Su^k - Ax^k)\|^2 \\ &= \|x^k - x^*\|^2 + \|\delta A^*(Su^k - Ax^k)\|^2 + 2\delta \langle x^k - x^*, A^*(Su^k - Ax^k) \rangle \\ &\leq \|x^k - x^*\|^2 + \delta^2 \|A^*\|^2 \|Su^k - Ax^k\|^2 + 2\delta \langle A(x^k - x^*), Su^k - Ax^k \rangle \\ &\leq \|x^k - x^*\|^2 + \delta^2 \|A\|^2 \|Su^k - Ax^k\|^2 - \delta \|u^k - Ax^k\|^2 - \delta \|Su^k - Ax^k\|^2 \\ &= \|x^k - x^*\|^2 - \delta(1 - \delta \|A\|^2) \|Su^k - Ax^k\|^2 - \delta \|u^k - Ax^k\|^2. \end{aligned} \quad (3.5)$$

Then, for $\delta \in \left(0, \frac{1}{\|A\|^2 + 1}\right)$, one has

$$\|y^k - x^*\| \leq \|x^k - x^*\| \quad \forall k. \quad (3.6)$$

From the nonexpansiveness of P_C , it follows that

$$\begin{aligned} \|z^k - x^*\| &= \|P_C((1 - \lambda_k \mu)y^k) - P_C(x^*)\| \leq \|(1 - \lambda_k \mu)y^k - x^*\| = \|(1 - \lambda_k \mu)y^k - (1 - \lambda_k \mu)x^* - \lambda_k \mu x^*\| \\ &\leq \|(1 - \lambda_k \mu)y^k - (1 - \lambda_k \mu)x^*\| + \lambda_k \mu \|x^*\| = (1 - \lambda_k \mu) \|y^k - x^*\| + \lambda_k \mu \|x^*\|. \end{aligned} \quad (3.7)$$

Combining $T(x^*) = x^*$, the nonexpansiveness of T , (3.7) and taking (3.6) into account, we obtain

$$\begin{aligned}
 \|x^{k+1} - x^*\| &= \|\alpha_k x^k + (1 - \alpha_k) T(z^k) - x^*\| = \|\alpha_k (x^k - x^*) + (1 - \alpha_k)(T(z^k) - x^*)\| \\
 &\leq \alpha_k \|x^k - x^*\| + (1 - \alpha_k) \|T(z^k) - x^*\| = \alpha_k \|x^k - x^*\| + (1 - \alpha_k) \|T(z^k) - T(x^*)\| \\
 &\leq \alpha_k \|x^k - x^*\| + (1 - \alpha_k) \|z^k - x^*\| \leq \alpha_k \|x^k - x^*\| + (1 - \alpha_k) \left[(1 - \lambda_k \mu) \|y^k - x^*\| + \lambda_k \mu \|x^*\| \right] \\
 &\leq \alpha_k \|x^k - x^*\| + (1 - \alpha_k) \left[(1 - \lambda_k \mu) \|x^k - x^*\| + \lambda_k \mu \|x^*\| \right] \\
 &= [1 - \lambda_k (1 - \alpha_k) \mu] \|x^k - x^*\| + (1 - \alpha_k) \lambda_k \mu \|x^*\|.
 \end{aligned}$$

In particular, $\|x^{k+1} - x^*\| \leq \max \{ \|x^k - x^*\|, \|x^*\| \}$, from which by induction follow that $\|x^k - x^*\| \leq \max \{ \|x^0 - x^*\|, \|x^*\| \} \forall k \geq 0$. Hence, the sequence $\{x^k\}$ is bounded, and so is $\{y^k\}$.

Using the nonexpansiveness of P_C , we get

$$\begin{aligned}
 \|y^{k+1} - y^k\|^2 &= \|P_C(x^{k+1} + \delta A^*(Su^{k+1} - Ax^{k+1})) - P_C(x^k + \delta A^*(Su^k - Ax^k))\|^2 \\
 &\leq \|x^{k+1} + \delta A^*(Su^{k+1} - Ax^{k+1}) - x^k - \delta A^*(Su^k - Ax^k)\|^2 \\
 &= \|(x^{k+1} - x^k) + \delta A^*(Su^{k+1} - Su^k + Ax^k - Ax^{k+1})\|^2 \\
 &= \|x^{k+1} - x^k\|^2 + \delta^2 \|A^*(Su^{k+1} - Su^k + Ax^k - Ax^{k+1})\|^2 \\
 &\quad + 2\delta \langle x^{k+1} - x^k, A^*(Su^{k+1} - Su^k + Ax^k - Ax^{k+1}) \rangle.
 \end{aligned} \tag{3.8}$$

Note that

$$\begin{aligned}
 \delta^2 \|A^*(Su^{k+1} - Su^k + Ax^k - Ax^{k+1})\|^2 &\leq \delta^2 \|A^*\|^2 \|Su^{k+1} - Su^k + Ax^k - Ax^{k+1}\|^2 \\
 &= \delta^2 \|A\|^2 \|Su^{k+1} - Su^k + Ax^k - Ax^{k+1}\|^2.
 \end{aligned} \tag{3.9}$$

Denoting $\Theta_k := 2\delta \langle x^{k+1} - x^k, A^*(Su^{k+1} - Su^k + Ax^k - Ax^{k+1}) \rangle$ and using the nonexpansiveness of S and P_Q , we obtain

$$\begin{aligned}
 \Theta_k &= 2\delta \langle A(x^{k+1} - x^k), Su^{k+1} - Su^k + Ax^k - Ax^{k+1} \rangle = 2\delta \langle Ax^{k+1} - Ax^k, Su^{k+1} - Su^k \rangle - 2\delta \|Ax^{k+1} - Ax^k\|^2 \\
 &= \delta \left[\|Su^{k+1} - Su^k\|^2 - \|Ax^{k+1} - Ax^k - (Su^{k+1} - Su^k)\|^2 - \|Ax^{k+1} - Ax^k\|^2 \right] \\
 &\leq \delta \left[\|u^{k+1} - u^k\|^2 - \|Ax^{k+1} - Ax^k - (Su^{k+1} - Su^k)\|^2 - \|Ax^{k+1} - Ax^k\|^2 \right] \\
 &= \delta \left[\|P_Q(Ax^{k+1}) - P_Q(Ax^k)\|^2 - \|Ax^{k+1} - Ax^k - (Su^{k+1} - Su^k)\|^2 - \|Ax^{k+1} - Ax^k\|^2 \right] \\
 &\leq -\delta \|Ax^{k+1} - Ax^k - (Su^{k+1} - Su^k)\|^2.
 \end{aligned} \tag{3.10}$$

Applying (3.9) and (3.10) to (3.8), and using the condition of δ , we have

$$\|y^{k+1} - y^k\|^2 \leq \|x^{k+1} - x^k\|^2 - \delta(1 - \delta\|A\|^2) \|Su^{k+1} - Su^k + Ax^k - Ax^{k+1}\|^2 \leq \|x^{k+1} - x^k\|^2.$$

Hence

$$\|y^{k+1} - y^k\| \leq \|x^{k+1} - x^k\| \quad \forall k \geq 0. \tag{3.11}$$

For simplicity of notation, let $t^k = T(z^k)$. From the nonexpansiveness of the mappings T and P_C and (3.11), we have

$$\begin{aligned}
 \|t^{k+1} - t^k\| &= \|T(z^{k+1}) - T(z^k)\| \leq \|z^{k+1} - z^k\| = \|P_C((1 - \lambda_{k+1}\mu)y^{k+1}) - P_C((1 - \lambda_k\mu)y^k)\| \\
 &\leq \|(1 - \lambda_{k+1}\mu)y^{k+1} - (1 - \lambda_k\mu)y^k\| = \|(1 - \lambda_k\mu)y^{k+1} - (1 - \lambda_k\mu)y^k + \mu(\lambda_k - \lambda_{k+1})y^{k+1}\| \\
 &\leq (1 - \lambda_k\mu)\|y^{k+1} - y^k\| + \mu|\lambda_k - \lambda_{k+1}|\|y^{k+1}\| \leq (1 - \lambda_k\mu)\|x^{k+1} - x^k\| + \mu|\lambda_k - \lambda_{k+1}|\|y^{k+1}\|.
 \end{aligned}$$

Thus, it follows from the last inequality that $\|t^{k+1} - t^k\| - \|x^{k+1} - x^k\| \leq -\lambda_k\mu\|x^{k+1} - x^k\| + \mu|\lambda_k - \lambda_{k+1}|\|y^{k+1}\|$. Since $\{x^k\}, \{y^k\}$ are bounded and $\lim_{k \rightarrow \infty} \lambda_k = 0$, we get $\limsup_{k \rightarrow \infty} (\|t^{k+1} - t^k\| - \|x^{k+1} - x^k\|) \leq 0$. So, utilizing

Lemma 2.2, we obtain $\lim_{k \rightarrow \infty} \|t^k - x^k\| = 0$.

Now observe that $\|x^{k+1} - x^k\| = (1 - \alpha_k)\|t^k - x^k\| \leq \|t^k - x^k\|$, and that

$$\begin{aligned}\|x^k - T(y^k)\| &\leq \|x^k - t^k\| + \|t^k - T(y^k)\| = \|x^k - t^k\| + \|T(z^k) - T(y^k)\| \leq \|x^k - t^k\| + \|z^k - y^k\| \\ &\leq \|x^k - t^k\| + \|P_C((1 - \lambda_k\mu)y^k) - P_C(y^k)\| \leq \|x^k - t^k\| + \|(1 - \lambda_k\mu)y^k - y^k\| \\ &= \|x^k - t^k\| + \lambda_k\mu\|y^k\|,\end{aligned}$$

which together with $\lim_{k \rightarrow \infty} \|t^k - x^k\| = 0$, $\lim_{k \rightarrow \infty} \lambda_k = 0$, by boundedness of $\{F(y^k)\}$, imply

$$\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0, \quad \lim_{k \rightarrow \infty} \|x^k - T(y^k)\| = 0. \quad (3.12)$$

Using successively the nonexpansiveness property of the mapping T and $T(x^*) = x^*$, we have, for all k ,

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &= \|\alpha_k(x^k - x^*) + (1 - \alpha_k)(T(z^k) - x^*)\|^2 \leq \alpha_k\|x^k - x^*\|^2 + (1 - \alpha_k)\|T(z^k) - x^*\|^2 \\ &= \alpha_k\|x^k - x^*\|^2 + (1 - \alpha_k)\|T(z^k) - T(x^*)\|^2 \leq \alpha_k\|x^k - x^*\|^2 + (1 - \alpha_k)\|z^k - x^*\|^2.\end{aligned} \quad (3.13)$$

From inequalities (3.5) and (3.7), we obtain

$$\begin{aligned}\|z^k - x^*\|^2 &\leq \left[(1 - \lambda_k\mu)\|y^k - x^*\| + \lambda_k\mu\|x^*\| \right]^2 \\ &= (1 - \lambda_k\mu)^2\|y^k - x^*\|^2 + \lambda_k\mu\|x^*\| \left[2(1 - \lambda_k\mu)\|y^k - x^*\| + \lambda_k\mu\|x^*\| \right] \\ &\leq (1 - \lambda_k\mu)^2 \left[\|x^k - x^*\|^2 - \delta(1 - \delta\|A\|^2)\|Su^k - Ax^k\|^2 - \delta\|u^k - Ax^k\|^2 \right] \\ &\quad + \lambda_k\mu\|x^*\| \left[2(1 - \lambda_k\mu)\|y^k - x^*\| + \lambda_k\mu\|x^*\| \right] \\ &\leq \|x^k - x^*\|^2 - \delta(1 - \lambda_k\mu)^2 \left[(1 - \delta\|A\|^2)\|Su^k - Ax^k\|^2 + \|u^k - Ax^k\|^2 \right] \\ &\quad + \lambda_k\mu\|x^*\| \left[2(1 - \lambda_k\mu)\|y^k - x^*\| + \lambda_k\mu\|x^*\| \right].\end{aligned} \quad (3.14)$$

Substituting (3.14) into (3.13) to deduce

$$\begin{aligned}\|x^{k+1} - x^*\|^2 &\leq \|x^k - x^*\|^2 - \delta(1 - \alpha_k)(1 - \lambda_k\mu)^2 \left[(1 - \delta\|A\|^2)\|Su^k - Ax^k\|^2 + \|u^k - Ax^k\|^2 \right] \\ &\quad + (1 - \alpha_k)\lambda_k\mu\|x^*\| \left[2(1 - \lambda_k\mu)\|y^k - x^*\| + \lambda_k\mu\|x^*\| \right].\end{aligned}$$

Using $\delta \in \left(0, \frac{1}{\|A\|^2 + 1}\right)$ and defining

$$\begin{aligned}nu_k &:= \delta(1 - \alpha_k)(1 - \lambda_k\mu)^2, \\ \psi &:= (1 - \alpha_k)\lambda_k\mu\|x^*\| \left[2(1 - \lambda_k\mu)\|y^k - x^*\| + \lambda_k\mu\|x^*\| \right],\end{aligned}$$

we get

$$\begin{aligned}v_k \left[(1 - \delta\|A\|^2)\|Su^k - Ax^k\|^2 + \|u^k - Ax^k\|^2 \right] &\leq (\|x^k - x^*\|^2 - \|x^{k+1} - x^*\|^2) + \psi_k \\ &\leq (\|x^k - x^*\| + \|x^{k+1} - x^*\|)\|x^{k+1} - x^k\| + \psi_k.\end{aligned} \quad (3.15)$$

Since $\lim_{k \rightarrow \infty} \|x^{k+1} - x^k\| = 0$, $\{x^k\}, \{y^k\}$ are bounded, $\lim_{k \rightarrow \infty} \lambda_k = 0$, $\lim_{k \rightarrow \infty} \alpha_k = \alpha \in (0, 1)$, we have $\lim_{k \rightarrow \infty} v_k = \delta(1 - \alpha) > 0$ and the right hand side of (3.15) tends to zero as $k \rightarrow \infty$. This implies that

$$\lim_{k \rightarrow \infty} \|Su^k - Ax^k\| = 0, \quad \lim_{k \rightarrow \infty} \|u^k - Ax^k\| = 0. \quad (3.16)$$

Then using again the fact that the projection operator P_C is nonexpansive, $\{x^k\} \subset C$, we can write

$$\begin{aligned}\|y^k - x^k\| &= \|P_C(x^k + \delta A^*(Su^k - Ax^k)) - P_C(x^k)\| \leq \|x^k + \delta A^*(Su^k - Ax^k) - x^k\| \\ &= \|\delta A^*(Su^k - Ax^k)\| \leq \delta\|A^*\|\|Su^k - Ax^k\| = \delta\|A\|\|Su^k - Ax^k\|,\end{aligned}$$

which together with (3.16) implies

$$\lim_{k \rightarrow \infty} \|y^k - x^k\| = 0. \quad (3.17)$$

From the triangle inequality, we get $\|y^k - T(y^k)\| \leq \|x^k - y^k\| + \|x^k - T(y^k)\|$, $\|u^k - Su^k\| \leq \|u^k - Ax^k\| + \|Su^k - Ax^k\|$, from which, by (3.17), (3.12) and (3.16) it follows that

$$\lim_{k \rightarrow \infty} \|y^k - T(y^k)\| = 0, \quad \lim_{k \rightarrow \infty} \|u^k - Su^k\| = 0. \quad (3.18)$$

Now we show that $\limsup_{k \rightarrow \infty} \langle x^*, x^* - y^k + \lambda_k \mu y^k \rangle \leq 0$. Take a subsequence $\{y^{k_i}\}$ of $\{y^k\}$ such that

$\limsup_{k \rightarrow \infty} \langle x^*, x^* - y^k \rangle = \lim_{i \rightarrow \infty} \langle x^*, x^* - y^{k_i} \rangle$. Since $\{y^{k_i}\}$ is bounded, we may assume that y^{k_i} converges weakly to $\bar{y}$. Therefore, $\limsup_{k \rightarrow \infty} \langle x^*, x^* - y^k \rangle = \lim_{i \rightarrow \infty} \langle x^*, x^* - y^{k_i} \rangle = \langle x^*, x^* - \bar{y} \rangle$. We observe that $\bar{y} \in C$ because $y^{k_i} \subset C$, $y^{k_i} \rightharpoonup \bar{y}$ and C is weakly closed.

Assume that $\bar{y} \notin \text{Fix}(T)$, that is $\bar{y} \neq T(\bar{y})$. Since $y^{k_i} \rightharpoonup \bar{y}$ and T is a nonexpansive mapping, from (3.18) and Opial's lemma, one has

$$\begin{aligned} \liminf_{i \rightarrow \infty} \|y^{k_i} - \bar{y}\| &< \liminf_{i \rightarrow \infty} \|y^{k_i} - T(\bar{y})\| \leq \liminf_{i \rightarrow \infty} [\|y^{k_i} - T(y^{k_i})\| + \|T(y^{k_i}) - T(\bar{y})\|] \\ &= \liminf_{i \rightarrow \infty} \|T(y^{k_i}) - T(\bar{y})\| \leq \liminf_{i \rightarrow \infty} \|y^{k_i} - \bar{y}\|. \end{aligned}$$

This is a contradiction. So $\bar{y} \in \text{Fix}(T)$.

Since $y^{k_i} \rightharpoonup \bar{y}$ and $\lim_{k \rightarrow \infty} \|y^k - x^k\| = 0$, we get $x^{k_i} \rightharpoonup \bar{y}$. Thus $Ax^{k_i} \rightharpoonup A\bar{y}$. Using this and (3.16), we have

$$u^{k_i} \rightharpoonup A\bar{y}. \quad (3.19)$$

Since $\{u^{k_i}\} \subset Q$ and Q is weakly closed, we derive by virtue of (3.19) that $A\bar{y} \in Q$.

Next we prove that $A\bar{y} \in \text{Fix}(S)$. Otherwise, if $S(A\bar{y}) \neq A\bar{y}$, and hence by Opial's lemma and (3.18), it turns out that

$$\begin{aligned} \liminf_{i \rightarrow \infty} \|u^{k_i} - A\bar{y}\| &< \liminf_{i \rightarrow \infty} \|u^{k_i} - S(A\bar{y})\| = \liminf_{i \rightarrow \infty} \|u^{k_i} - Su^{k_i} + Su^{k_i} - S(A\bar{y})\| \\ &\leq \liminf_{i \rightarrow \infty} (\|u^{k_i} - Su^{k_i}\| + \|Su^{k_i} - S(A\bar{y})\|) = \liminf_{i \rightarrow \infty} \|Su^{k_i} - S(A\bar{y})\| \leq \liminf_{i \rightarrow \infty} \|u^{k_i} - A\bar{y}\|, \end{aligned}$$

which leads to a contradiction. Therefore $A\bar{y} \in \text{Fix}(S)$.

Since $\bar{y} \in \text{Fix}(T)$ and $A\bar{y} \in \text{Fix}(S)$, we obtain $\bar{y} \in \Omega$. It then follows from $x^* \in \text{Sol}(\Omega)$ that $\langle x^*, \bar{y} - x^* \rangle \geq 0$. Thus, the boundedness of $\{y^k\}$ and $\lim_{k \rightarrow \infty} \lambda_k = 0$ yield

$$\begin{aligned} \limsup_{k \rightarrow \infty} \langle x^*, x^* - y^k + \lambda_k \mu y^k \rangle &= \limsup_{k \rightarrow \infty} [\langle x^*, x^* - y^k \rangle + \lambda_k \mu \langle x^*, y^k \rangle] \\ &= \limsup_{k \rightarrow \infty} \langle x^*, x^* - y^k \rangle = \langle x^*, x^* - \bar{y} \rangle \leq 0. \end{aligned}$$

Finally, we prove that x^k converges to the point x^* . Using the nonexpansiveness of P_C , the inequality $\|x - y\|^2 \leq \|x\|^2 - 2\langle y, x - y \rangle$ valid for any $x, y \in \mathcal{H}_1$, by (3.6), we obtain successively

$$\begin{aligned} \|z^k - x^*\|^2 &= \|P_C((1 - \lambda_k \mu)y^k) - P_C(x^*)\|^2 \leq \|(1 - \lambda_k \mu)y^k - x^*\|^2 = \|(1 - \lambda_k \mu)y^k - (1 - \lambda_k \mu)x^* - \lambda_k \mu x^*\|^2 \\ &\leq \|(1 - \lambda_k \mu)y^k - (1 - \lambda_k \mu)x^*\|^2 - 2\lambda_k \mu \langle x^*, (1 - \lambda_k \mu)y^k - x^* \rangle \\ &= (1 - \lambda_k \mu)^2 \|y^k - x^*\|^2 - 2\lambda_k \mu \langle x^*, (1 - \lambda_k \mu)y^k - x^* \rangle \\ &\leq (1 - \lambda_k \mu)^2 \|x^k - x^*\|^2 - 2\lambda_k \mu \langle x^*, (1 - \lambda_k \mu)y^k - x^* \rangle. \end{aligned}$$

Substituting this inequality into (3.13), we have

$$\begin{aligned} \|x^{k+1} - x^*\|^2 &\leq \alpha_k \|x^k - x^*\|^2 + (1 - \alpha_k) \|z^k - x^*\|^2 \\ &\leq \alpha_k \|x^k - x^*\|^2 + (1 - \alpha_k) (1 - \lambda_k \mu) \|x^k - x^*\|^2 - 2\lambda_k \mu (1 - \alpha_k) \langle x^*, (1 - \lambda_k \mu)y^k - x^* \rangle \\ &= [1 - \lambda_k (1 - \alpha_k) \mu] \|x^k - x^*\|^2 + \lambda_k (1 - \alpha_k) \mu \theta_k, \end{aligned} \quad (3.20)$$

where $\theta_k = 2 \langle x^*, x^* - y^k + \lambda_k \mu y^k \rangle$. Since $\limsup_{k \rightarrow \infty} \langle x^*, x^* - y^k + \lambda_k \mu y^k \rangle \leq 0$, we get $\limsup_{k \rightarrow \infty} \theta_k \leq 0$. Note that $\sum_{k=0}^{\infty} \lambda_k (1 - \alpha_k) \tau = \infty$ due to condition 3.1, an application of Lemma 2.4 to (3.20) yields $x^k \rightarrow x^*$, which completes the proof. $\square$

Remark 3.1. In Theorem 3.1, we can choose, for example, $\lambda_k = \frac{1}{k+3}$, $\alpha_k = \frac{k+2}{3(k+1)}$. An elementary computation shows that $\{\lambda_k\} \subset (0, 1)$, $\{\alpha_k\} \subset (0, 1)$. Furthermore, $\lim_{k \rightarrow \infty} \lambda_k = 0$, $\sum_{k=0}^{\infty} \lambda_k (1 - \alpha_k) = \infty$, $\lim_{k \rightarrow \infty} \alpha_k = \frac{1}{3} \in (0, 1)$. Thus conditions (a)–(c) in Theorem 3.1 are satisfied.

4 Numerical Results

In the following example, we immediately apply (3.1) to approximate the solution of problem and compare it with the exact solution. We perform the iterative schemes in Python running on a laptop with Intel(R) Core(TM) i5-5200U CPU @ 2.20 GHz, RAM 12 GB.

Example 4.1. Consider (MNPP)–(SFP) with the following suppositions: The Hilbert spaces $\mathcal{H}_1 = \mathbb{R}^2$, $\mathcal{H}_2 = \mathbb{R}^3$. Let $C = \{x \in \mathbb{R}^2 \mid 2x_1 + x_2 \leq 0\}$ and $Q = \{x \in \mathbb{R}^3 \mid 2x_1 + x_2 - x_3 + 1 = 0\}$ be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The linear operator $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$, defined as $Ax = (x_1 + 2x_2, 3x_1 - x_2, x_2)$, with the adjoint operator $A^* : \mathcal{H}_2 \rightarrow \mathcal{H}_1$, which defined as $A^*y = (y_1 + 3y_2, 2y_1 - y_2 + y_3)$. The nonexpansive mappings $T : C \rightarrow C$ and $S : Q \rightarrow Q$ are respectively defined as: $T(x) = P_C(x)$, $S(x) = P_Q(x)$. Then the bilevel problem (MNPP)–(SFP) become:

$$\text{Find } x^* = \underset{\Omega}{\operatorname{argmin}} \|x\|, \quad (\text{MNPP}_1)$$

where Ω is the solution set of the split fixed point problem:

$$\text{find } x^* \in \operatorname{Fix}(P_C) \text{ such that } Ax^* \in \operatorname{Fix}(P_Q). \quad (\text{SFP}_1)$$

Since P_C and P_Q are nonexpansive, and $\operatorname{Fix}(P_C) = C$, $\operatorname{Fix}(P_Q) = Q$, then we can find easily that the solution set of (SFP₁) is the ray $\Omega = \{x \in \mathbb{R}^2 \mid 5x_1 + 2x_2 + 1 = 0, x_1 \geq -1\}$, and the unique solution of (MNPP₁) is $x^* = \left(-\frac{5}{29}, -\frac{2}{29}\right)^\top$, which is the metric projection of $O = (0, 0)^\top$ onto Ω .

We now consider the convergence of iterative method (3.1) for Example 4.1. We obtain the following Tables 1, 2 and 3 of numerical results.

Applying (3.1) with $\delta = \frac{1}{2}$, $\mu = 2$, $\lambda_k = \frac{1}{k+2}$, $\alpha_k = \frac{k+1}{2(k+3)}$, starting point $x^0 = (0, 0)^\top$, stop condition $\|x^k - x^*\| \leq 10^{-6}$, we get Table 1 of numerical results.

Applying (3.1) with $\delta = \frac{1}{2}$, $\mu = 2$, $\lambda_k = \frac{1}{1.7k+2}$, $\alpha_k = \frac{k^{0.01}+1}{2(k^{0.01}+3)}$, starting point $x^0 = (0, 0)^\top$, stop condition $\|x^k - x^*\| \leq 10^{-6}$, we get Table 2 of numerical results.

Applying (3.1) with $\delta = \frac{1}{2}$, $\mu = 2$, $\lambda_k = \frac{1}{9k+9}$, $\alpha_k = \frac{e^{k^{0.6}}+1}{2(e^{k^{0.6}}+3)}$, starting point $x^0 = (0, 0)^\top$, stop condition $\|x^k - x^*\| \leq 10^{-6}$, we get Table 3 of numerical results.

In all of cases, the sequence $\{x^k\}$ converges to $x^* = \left(-\frac{5}{29}, -\frac{2}{29}\right)^\top \approx (-0.17241379, -0.06896551)^\top$.

Table 1. $x^0 = (0, 0)^\top$, $\delta = \frac{1}{2}$, $\mu = 2$, $\lambda_k = \frac{1}{k+2}$, $\alpha_k = \frac{k+1}{2(k+3)}$

Iter (k)	x_1^k	x_2^k	$\ x^k - x^{k-1}\ $	$\ x^k - x^*\ $	Time (s)
50	-0.1695609	-0.06782436	6.267090×10^{-5}	0.003072	0.01561
500	-0.17212842	-0.06885137	6.159414×10^{-7}	0.000307	0.04689
100000	-0.17241237	-0.06896495	1.536815×10^{-11}	1.536789×10^{-6}	5.68423
153678	-0.17241286	-0.06896515	6.507027×10^{-12}	9.999993×10^{-7}	8.845065

Table 2. $x^0 = (0,0)^\top$, $\delta = \frac{1}{2}$, $\mu = 2$, $\lambda_k = \frac{1}{1.7k+2}$, $\alpha_k = \frac{k^{0.01} + 1}{2(k^{0.01} + 3)}$

Iter (k)	x_1^k	x_2^k	$\ x^k - x^{k-1}\ $	$\ x^k - x^*\ $	Time (s)
50	-0.17073279	-0.06829312	3.701215×10^{-5}	0.001810	0.0
500	-0.1722459	-0.06889836	3.624207×10^{-7}	0.000180	0.03124
10000	-0.1724054	-0.06896216	9.040966×10^{-10}	9.039997×10^{-6}	0.57810
90399	-0.17241286	-0.06896515	1.106215×10^{-11}	9.999936×10^{-7}	5.25717

Table 3. $x^0 = (0,0)^\top$, $\delta = \frac{1}{2}$, $\mu = 2$, $\lambda_k = \frac{1}{9k+9}$, $\alpha_k = \frac{e^{k^{0.6}} + 1}{2(e^{k^{0.6}} + 3)}$

Iter (k)	x_1^k	x_2^k	$\ x^k - x^{k-1}\ $	$\ x^k - x^*\ $	Time (s)
50	-0.17209698	-0.06883879	6.957341×10^{-6}	0.000341	0.01560
500	-0.17238209	-0.06895283	6.842703×10^{-8}	3.414798×10^{-5}	0.04685
10000	-0.17241221	-0.06896488	1.707698×10^{-10}	1.707536×10^{-6}	0.59370
17075	-0.17241286	-0.06896515	5.856303×10^{-11}	9.999643×10^{-7}	1.01555

5 Conclusion

We have proposed a strongly convergent algorithm for the minimum norm point in the solution set of split fixed point problems. The proposed algorithm is a combination between the projection method commonly used for solving variational inequality problems with the well-known Krasnoselskii–Mann iterative scheme for finding fixed point of nonexpansive mappings. As a sequence, we have obtained algorithms for finding the minimum norm point in the solution set of split variational inequality problems.

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