

THE MALLIAVIN DERIVATIVE FOR GENERALIZATION OF FRACTIONAL BESSEL TYPE PROCESS

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ARTICLE INFO	ABSTRACT
Received: 07/4/2021 Revised: 28/5/2021 Published: 31/5/2021	This paper considers a generalization of fractional Bessel type process. It is also a type of singular stochastic differential equations driven by fractional Brownian motion which were studied by some authors. Under some assumptions of coefficients, this equation has a unique positive solution. The main purpose of this paper is to show the formula of the Malliavin derivative for this process. The techniques of Malliavin calculus were applied for stochastic differential equations driven by a fractional Brownian motion. We obtain that the Malliavin derivative for this process is an exponent function of the drift coefficient's derivative. This result is useful to estimate inverse moments of the solution. From that, we can estimate the rate of convergence of the numerical approximation in the L_p- norm.
KEYWORDS Fractional Brownian motion Fractional Bessel process Fractional stochastic differential equation Malliavin derivative Malliavin calculus	

ĐẠO HÀM MALLIAVIN CỦA QUÁ TRÌNH BESSEL PHÂN THỨ DẠNG TỔNG QUÁT

Vũ Thị Hương

Trường Đại học Giao thông Vận tải

THÔNG TIN BÀI BÁO	TÓM TẮT
Ngày nhận bài: 07/4/2021 Ngày hoàn thiện: 28/5/2021 Ngày đăng: 31/5/2021	Bài báo này xem xét một dạng tổng quát của quá trình Bessel phân thứ. Đây cũng là một dạng thuộc lớp các phương trình vi phân ngẫu nhiên kỳ dị xác định bởi chuyển động Brown phân thứ đã được nghiên cứu bởi một số tác giả. Với một số giả thiết của các hệ số, phương trình này có nghiệm duy nhất dương. Mục đích chính của bài báo là đưa ra công thức của đạo hàm Malliavin cho quá trình này. Tính toán Malliavin được sử dụng cho phương trình vi phân ngẫu nhiên xác định bởi chuyển động Brown phân thứ. Chúng ta nhận được đạo hàm Malliavin cho quá trình này là một hàm mũ của đạo hàm của hệ số dịch chuyển. Kết quả này rất hữu ích khi đánh giá moment ngược của nghiệm. Từ đó chúng ta có thể đánh giá tốc độ hội tụ của nghiệm xấp xỉ trong L_p.
TỪ KHÓA Chuyển động Brown phân thứ Quá trình Bessel phân thứ Phương trình vi phân ngẫu nhiên phân thứ Đạo hàm Malliavin Tính toán Malliavin	

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1 Introduction

In [1], the authors considered the following singular stochastic differential equation driven by fractional Brownian motion with Hurst parameter $H > 1/2$.

$$X_t = x_0 + \int_0^t \frac{K}{X_s} ds + B_t^H, \quad (1)$$

where $K > 0$ is a constant. They showed the formula for the Malliavin derivative of the solution and then estimated negative moments. In [2], Mishura and Yurchenko-Tytarenko studied the fractional stochastic differential equation

$$dY(t) = \frac{1}{2} \left(\frac{k}{Y(t)} - aY(t) \right) dt + \frac{1}{2} \sigma dB_t^H, \quad Y(0) > 0, k > 0, \quad (2)$$

They considered the probability of hitting zero of the solution. This equation was also studied in [3]. Authors introduced an implicit Euler scheme and showed that the scheme strongly converges at the rate of order 1. They also gave the boundedness of inverse moments of the numerical solution and derived the Malliavin derivative of the numerical solution. In this paper, we consider a more general singular stochastic differential equation driven by fractional Brownian motion. More precisely, we study a generalization of Bessel type process $Y = (Y(t))_{0 \leq t \leq T}$ satisfying the following SDEs,

$$dY(t) = \left(\frac{k}{Y(t)} + b(t, Y(t)) \right) dt + \sigma dB^H(t), \quad (3)$$

where $0 \leq t \leq T$, $Y(0) > 0$ and B^H is a fractional Brownian motion with the Hurst parameter $H > \frac{1}{2}$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t, t \in [0, T]\}$ satisfying the usual condition. Fix $T > 0$ and we consider equation (3) on the interval $[0, T]$. We suppose that $k > 0$ and the coefficient $b = b(t, x) : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function and globally Lipschitz continuous with respect to x , linearly growth with respect to x . It means that there exists positive constants L, C such that the following conditions hold:

$$A_1) \quad |b(t, x) - b(t, y)| \leq L|x - y|, \text{ for all } x, y \in \mathbb{R} \text{ and } t \in [0, T];$$

$$A_2) \quad |b(t, x)| \leq C(1 + |x|), \text{ for all } x \in \mathbb{R} \text{ and } t \in [0, T];$$

$$A_3) \quad b(t, x) \text{ has a continuous partial derivative with respect to } x.$$

The existence and uniqueness of solution for generalization of the fractional Bessel type process was proved in [4] by the following Theorems

Theorem 1.1. *Assume that conditions $(A_1) - (A_2)$ are satisfied. Then for each $T > 0$ equation (3) has a unique solution on $[0, T]$.*

Lemma 1.2. *Let $(Y(t))_{0 \leq t \leq T}$ be the solution of equation (3) then $Y(t) > 0$ for all $t \in [0, T]$ almost surely.*

The main purpose of this paper is to study the Malliavin derivative of the solution using Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion. So, firstly, we recall some basic facts on Malliavin calculus (see [5], [6], [7]).

2 Malliavin calculus

Fix a time interval $[0, T]$. We consider a fractional Brownian motion $\{B^H(t)\}_{t \in [0, T]}$. We note that $E(B^H(s).B^H(t)) = R_H(s, t)$ where

$$R_H(s, t) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}). \quad (4)$$

We denote by $\mathcal{E}$ the set of real valued step functions on $[0, T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle I_{[0, t]}, I_{[0, s]} \rangle_{\mathcal{H}} = R_H(t, s). \quad (5)$$

On the other hand, in [6], authors showed that the covariance $R_H(t, s)$ can be written as

$$R_H(t, s) = \alpha_H \int_0^t \int_0^s |r - u|^{2H-2} du dr = \int_0^{t \wedge s} K_H(t, r) K_H(s, r) dr, \quad (6)$$

where $\alpha_H = H(2H - 1)$, $K_H(t, s)$ is the square integrable kernel defined by

$$K_H(t, s) = c_H s^{\frac{1}{2}-H} \int_0^t (u - s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad (7)$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and β denote the Beta function. We put $K_H(t, s) = 0$ if $t \leq s$.

It implies that for any pair of step functions $\varphi, \psi \in \mathcal{E}$ we have

$$\langle \varphi, \psi \rangle = \alpha_H \int_0^T \int_0^T |r - u|^{2H-2} \varphi_r \psi_u du dr \quad (8)$$

The mapping $I_{[0, t]} \mapsto B^H(t)$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space associated with B^H . Denote this isometry by $\varphi \mapsto B^H(\varphi)$.

Definition 2.1. *Let $\mathcal{S}$ be the space of smooth and cylindrical random variables of the form*

$$F = f(B^H(\varphi_1), \dots, B^H(\varphi_n)),$$

where $n \geq 1, f \in C_b^\infty(\mathbb{R}^n)$. We define the derivative operator DF on $F \in \mathcal{S}$ as the $\mathcal{H}$ -valued random variable

$$DF = \sum_{i=1}^n \frac{\partial F}{\partial x_i}(B^H(\varphi_1), \dots, B^H(\varphi_n))\varphi_i. \quad (9)$$

We call the derivative operator DF as the Malliavin derivative of random variable F .

We consider the linear operator $K_H^* : \mathcal{E} \rightarrow L^2([0, T])$ defined by

$$(K_H^*\varphi)(s) = \int_s^T \varphi(t) \frac{\partial K_H}{\partial t}(t, s) dt. \quad (10)$$

Notice that $(K_H^*(1_{[0,t]}))(s) = K_H(t, s)1_{[0,t]}(s)$. Then for any $\varphi, \phi \in \mathcal{E}$ we have

$$\langle \varphi, \phi \rangle_{\mathcal{H}} = \langle K_H^*\varphi, K_H^*\phi \rangle_{L^2([0,T])} = E(B^H(\varphi)B^H(\phi)). \quad (11)$$

Then K_H^* provides an isometry between the Hilbert space $\mathcal{H}$ and a closed subspace of $L^2([0, T])$.

We define the operator $\mathcal{K}_H : L^2([0, T]) \rightarrow \mathcal{H}_H = \mathcal{K}_H(L^2([0, T]))$

$$(\mathcal{K}_H h)(t) = \int_0^t K_H(t, s)h(s)ds. \quad (12)$$

We denote by $\mathcal{R}_H = \mathcal{K}_H \circ K_H^* : \mathcal{H} \rightarrow \mathcal{H}_H$ the operator defined as following

$$\mathcal{R}_H \varphi = \int_0^\cdot K_H(\cdot, s)(K_H^*\varphi)(s)ds. \quad (13)$$

In [7], authors proved that for any $\varphi \in \mathcal{H}$, $\mathcal{R}_H \varphi$ is Holder continuous of order H .

The following Lemma is an integration by parts formula (see Lemma 11 in [7])

Lemma 2.2. *Let λ such that $\lambda + H > 1$, $f \in C^\lambda(0, T)$ and $h \in \mathcal{H}$. Then it holds that*

$$\int_0^T f(r)d(\mathcal{R}_H \varphi)(r) = \int_0^T f(r) \left(\int_0^r \frac{\partial K_H}{\partial r}(r, t)(K_H^*\varphi(t)dt) \right) dr. \quad (14)$$

Let $(Y(t))_{t \in [0, T]}$ be the solution of equation (3).

Definition 2.3. *The derivative defined by*

$$D_{\mathcal{R}_H \varphi} Y(t) = \frac{d}{d\epsilon} Y(t)(\omega + \epsilon \mathcal{R}_H \varphi)|_{\epsilon=0}, \quad (15)$$

exists for all $\varphi \in \mathcal{H}$ (see [7]).

Lemma 2.4. *The derivative $D_{\mathcal{R}_H \varphi} Y(t)$ coincides with $\langle DY(t), \varphi \rangle_{\mathcal{H}}$, where DY is the Malliavin derivative of Y (see [7]).*

3 Estimate the Malliavin derivative of solution

The main result of this paper provides an estimation for the Malliavin derivative of solution $Y(t)$ of equation (3) which is stated as the following Theorem

Theorem 3.1. *Assume that conditions $(A_1) - (A_3)$ are satisfied and $Y(t)$ is the solution of equation (3), then for any $t > 0$, we have*

$$DY(t) = \sigma \cdot \exp \left(\int_0^t \left[\frac{k}{Y^2(r)} + \frac{\partial b}{\partial y}(r, Y(r)) \right] dr \right) \mathbf{1}_{[0,t]}. \quad (16)$$

Proof. Fix a time interval $[0, T]$. Firstly, we compute $\langle DY(t), \varphi \rangle_{\mathcal{H}}, \varphi \in \mathcal{H}$. Setting $h = \mathcal{R}_H \varphi$. By equation (14) we have

$$\langle DY(t), \varphi \rangle_{\mathcal{H}} = D_{\mathcal{R}_H \varphi} Y(t) = \frac{dY^\epsilon(t)}{d\epsilon} \Big|_{\epsilon=0} = 0, \quad (17)$$

where $Y^\epsilon(t)$ is the solution of the equation

$$Y^\epsilon(t) = Y(0) + \int_0^t \left[\frac{k}{Y^\epsilon(s)} + b(s, Y^\epsilon(s)) \right] ds + \sigma B^H(t) + \epsilon \cdot \sigma h(t), t \in [0, T], \epsilon \in [0, 1]. \quad (18)$$

It leads to

$$Y^\epsilon(t) - Y(t) = \int_0^t \left[\frac{k}{Y^\epsilon(s)} + b(s, Y^\epsilon(s)) - \left(\frac{k}{Y(s)} + b(s, Y(s)) \right) \right] ds + \epsilon \cdot \sigma h(t). \quad (19)$$

From assumption A_3 , using Taylor expansion, we have

$$Y^\epsilon(t) - Y(t) = \int_0^t \partial_y f(s, Y(s) + \theta_s^\epsilon(Y^\epsilon(s) - Y(s)))(Y^\epsilon(s) - Y(s)) ds + \epsilon \cdot \sigma h(t), \quad (20)$$

where $f = f(t, y) = \frac{k}{y} + b(t, y)$, θ_s^ϵ between 0 and 1. This equation is a first order linear partial differential equation. Then the formula of the solution of equation is that

$$Y^\epsilon(t) - Y(t) = \epsilon \cdot \sigma \int_0^t \exp \left(\int_s^t \partial_y f(r, Y(r) + \theta_r^\epsilon(Y^\epsilon(r) - Y(r))) dr \right) dh(s). \quad (21)$$

Using Lemma (2.2), we have

$$\begin{aligned} Y^\epsilon(t) - Y(t) &= \epsilon \cdot \sigma \int_0^t \exp \left(\int_s^t \partial_y f(r, Y(r) + \theta_r^\epsilon(Y^\epsilon(r) - Y(r))) dr \right) \left(\int_0^s \frac{\partial K_H}{\partial s}(s, u) (K_H^* \varphi)(u) du \right) ds \\ &= \epsilon \cdot \sigma \int_0^t \left(\int_u^t \exp \left(\int_s^t \partial_y f(r, Y(r) + \theta_r^\epsilon(Y^\epsilon(r) - Y(r))) dr \right) \frac{\partial K_H}{\partial s}(s, u) ds \right) (K_H^* \varphi)(u) du. \end{aligned} \quad (22)$$

By the definition of the operator K_H^* we have

$$\begin{aligned} Y^\epsilon(t) - Y(t) &= \epsilon \cdot \sigma \int_0^t (K_H^* \varphi)(u) \left(K_H^* \left(\exp \left(\int_u^t \partial_y f(r, Y(r) + \theta_r^\epsilon(Y^\epsilon(r) - Y(r))) dr \right) \right) \right) (u) du \\ &= \epsilon \cdot \sigma \langle \varphi, \exp \left(\int_0^t \partial_y f(r, Y(r) + \theta_r^\epsilon(Y^\epsilon(r) - Y(r))) dr \right) \rangle_{\mathcal{H}} \\ &= \alpha_H \epsilon \cdot \sigma \int_0^t \int_0^t \varphi(s) \exp \left(\int_u^t \partial_y f(r, Y(r) + \theta_r^\epsilon(Y^\epsilon(r) - Y(r))) dr \right) |s - u|^{2H-2} du ds. \end{aligned}$$

By assumption that $k > 0$ we have

$$\partial_y f(t, y) = \frac{-k}{y^2} + \partial_y b(t, y) \leq \partial_y b(t, y).$$

But from assumption A_1 ,

$$\frac{b(t, x) - b(t, y)}{x - y} \leq \frac{|b(t, x) - b(t, y)|}{|x - y|} \leq L.$$

Then $\partial_y b(t, y) \leq L$. It mean that $\partial_y f(t, y)$ is bounded. As a consequence,

$$\begin{aligned} \lim_{\epsilon} \frac{Y^\epsilon(t) - Y(t)}{\epsilon} &= \alpha_H \cdot \sigma \int_0^t \int_0^t \varphi(s) \exp \left(\int_u^t \partial_y f(r, Y(r)) dr \right) |s - u|^{2H-2} du ds \\ &= \sigma \langle \varphi, \exp \left(\int_0^t \partial_y f(r, Y(r)) dr \right) \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}}, \end{aligned}$$

where the limit holds almost surely and in $L^2(\Omega)$.

Together with equation (16), it leads to

$$\begin{aligned} DY(t) &= \sigma \cdot \exp \left(\int_0^t \partial_y f(r, Y(r)) dr \right) \mathbf{1}_{[0,t]} \\ &= \sigma \cdot \exp \left(\int_0^t \left[\frac{k}{Y^2(r)} + \frac{\partial b}{\partial y}(r, Y(r)) \right] dr \right) \mathbf{1}_{[0,t]}. \end{aligned}$$

We obtain the formula of the Malliavin derivative DY .

4 Conclusion

The main result of this paper is to prove the formula of the Malliavin derivative for a type of singular process driven fractional Brownian motion. From this formula, in the future, we can estimate inverse moments of this process which is necessary in showing the rate convergence of the numerical approximation in the L_p - norm.

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