WEAK AND STRONG CONVERGENCE THEOREMS OF ALGORITHMIC SCHEMES FOR THE MULTIPLE-SETS SPLIT FEASIBILITY PROBLEM

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ABSTRACT

In this paper, for solving the multiple-sets split feasibility problem (MSSFP) in Hilbert spaces, we present a general approach to design iterative methods. We propose a weak convergence string-averaged algorithmic scheme and a strong convergence string-averaged algorithmic scheme. The strong convergence string averaged algorithmic scheme is constructed based on the general iterative method for nonexpansive mappings, in which the step-size is computed directly in each iteration without the knowledge of the operator norm. These algorithmic schemes contain not only improvement modifications of the well known cyclic and simultaneous iterative methods as particular cases but also new ones.

CÁC ĐỊNH LÝ HỘI TỤ YẾU VÀ HỘI TỤ MẠNH CỦA CÁC LƯỢC ĐỒ THUẬT TOÁN CHO BÀI TOÁN CHẤP NHẬN TÁCH ĐA TẬP

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Lược đồ thuật toán xâu trung bình

TÓM TẮT

Trong bài báo này, để giải bài toán chấp nhận tách đa tập (MSSFP) trong không gian Hilbert, chúng tôi trình bày một cách tiếp cận tổng quát để xây dựng các phương pháp lặp. Chúng tôi đề xuất một lược đồ thuật toán xâu trung bình với sự hội tụ yếu và một lược đồ thuật toán xâu trung bình với sự hội tụ mạnh. Lược đồ thuật toán xâu trung bình với sự hội tụ mạnh được xây dựng dựa trên phương pháp lặp tổng quát cho ánh xạ không giãn, trong đó cỡ bước được tính toán trực tiếp trong mỗi bước lặp mà không cần sử dụng chuẩn của toán tử. Những lược đồ thuật toán này không chỉ bao hàm những cải tiến của phương pháp lặp vòng và lặp đồng thời đã biết như những trường hợp riêng mà còn bao hàm cả những phương pháp lặp mới.

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1. Introduction and preliminaries

Let $I = \{1, 2, \dots, N\}$ and $J = \{1, 2, \dots, M\}$ with any fixed positive integers N and M . Let H_1 and H_2 be two real Hilbert spaces with $\{C_i\}_{i \in I}$ and $\{Q_j\}_{j \in J}$ being two families of closed convex subsets in H_1 and H_2 , respectively. Let A be a bounded linear mapping from H_1 into H_2 with the norm denoted by $\|A\|$. We use the symbol E , $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ to define the identity mapping, an inner product and a norm, respectively, in any Hilbert space.

The MSSFP is to find a point

$$p \in C := \bigcap_{i \in I} C_i \quad \text{such that} \quad Ap \in Q := \bigcap_{j \in J} Q_j. \quad (1.1)$$

Denote by Γ the set of solutions for (1.1). Throughout this paper, we assume that $\Gamma \neq \emptyset$. This problem was first introduced by Censor and Elfving [1] for modeling inverse problems that arise from phase retrievals and in image reconstruction [2], [3]. Recently, the MSSFP can also be used to model the intensity-modulated radiation therapy [4], [5].

For solving the split feasibility problem, that is (1.1) with $N = M = 1$, Byrne [2], [3] introduced a well-known iterative method, named CQ-method and defined by

$$x^{k+1} = P_C(E - \gamma A^T(E - P_Q)A)x^k, \quad k \geq 1, \quad (1.2)$$

with a fixed real number $\gamma \in (0, 2/\|A\|^2)$, where P_C and P_Q denote the metric projections on the sets C and Q , respectively, and A^T stands for the transpose of A .

Up to now, there is a long list of works concerning iterative methods to solve (1.1), for example, see [6]–[13] and references therein. These methods are extensions and modifications of (1.2) in different directions. In many works, ones used the gradient of a proximity function, that measures the distance of a point to all the sets in the image space, to construct iterative methods. Among them, there are two important ones, the first of which is the cyclic iterative method,

$$x^{k+1} = P_{C_{m(k)}}(E - \gamma \sum_{j=1}^M \eta_j A^*(E - P_{Q_j})A)x^k, \quad (1.3)$$

where $m(k) = k \bmod (N+1)$, $\eta_j > 0$ for every $1 \leq j \leq M$ and A^* denotes the adjoint of A . Another method is the simultaneous iterative one,

$$x^{k+1} = \sum_{i=1}^N \beta_i P_{C_i}(E - \gamma \sum_{j=1}^M \eta_j A^*(E - P_{Q_j})A)x^k, \quad (1.4)$$

where $\beta_i > 0$ for all i such that $\sum_{i=1}^N \beta_i = 1$ and $0 < \gamma < 2/\tilde{L}$ with $\tilde{L} = \|A\|^2 \sum_{j=1}^M \eta_j$. Xu [10] showed that (1.3) and (1.4) converge weakly to an element in Γ . Recently, Buong [6] proposed two weakly convergent improvement modifications of (1.3) and (1.4) with two new iterative methods, described in a general form,

$$x^{k+1} = P_1(E - \gamma A^*(E - P_2)A)x^k, \quad (1.5)$$

where $P_1 = P_{C_1} \cdots P_{C_N}$ or $P_1 = \sum_{i=1}^N \beta_i P_{C_i}$ and $P_2 = P_{Q_1} \cdots P_{Q_M}$ or $P_2 = \sum_{j=1}^M \eta_j P_{Q_j}$, with positive real numbers β_i and η_j , satisfying $\sum_{i=1}^N \beta_i = \sum_{j=1}^M \eta_j = 1$.

In the case that $H_1 \equiv H_2$ and $A = E$, the MSSFP deduces to the convex feasibility problem (CFP), that is to find a point $p \in C$. To solve the CFP, Censor et al. [14] proposed an algorithmic scheme in which the end-points of strings of sequential projections onto the constraints are averaged.

In Section 2, motivated by the results in the works above, we introduce two algorithmic schemes that contains the algorithm (1.5) and new others as particular cases. Specifically, in Section 2.1, for solving (1.1), we propose a new weak convergence string-averaged algorithmic scheme. In Section 2.2, in order to obtain the strong convergence, based on the general iterative method for nonexpansive mappings [15], we give a new general string-averaged algorithmic scheme with a self-adaptive step-size.

First, we list some facts that will be used in the proof of our results.

Definition 1.1 A mapping T from a subset C of a Hilbert space H into H is called:

- (i) θ -Lipschitz continuous with a Lipschitz constant $\theta \in [0, \infty)$ if $\|Tx - Ty\| \leq \theta\|x - y\|$ for all $x, y \in C$ and nonexpansive if $\theta = 1$;
- (ii) γ -inverse strongly monotone if $\gamma\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle$ for all $x, y \in C$ where γ is a positive number and firmly nonexpansive if $\gamma = 1$;
- (iii) averaged, if $T = (1 - \alpha)E + \alpha U$ for some fixed $\alpha \in (0, 1)$ and a nonexpansive mapping U and we say T is α -averaged;
- (iv) η -strongly monotone and γ -strictly pseudocontractive, if it satisfies, respectively, the conditions, $\langle Tx_1 - Tx_2, x_1 - x_2 \rangle \geq \eta\|x_1 - x_2\|^2$ and $\langle Tx_1 - Tx_2, x_1 - x_2 \rangle \leq \|x_1 - x_2\|^2 - \gamma\|(E - T)x_1 - (E - T)x_2\|^2$, for all $x_1, x_2 \in C$, η and γ are some positive real numbers.

For a closed convex subset C of H , there exists a mapping P_C from H onto C such that $\|P_C x - x\| = \inf_{y \in C} \|y - x\|$ for each $x \in H$. The mapping P_C is called the metric projection on C . We know that P_C is firmly nonexpansive (hence, nonexpansive) and $(1/2)$ -averaged. Moreover, $\|x - P_C x\|^2 + \|P_C x - z\|^2 \leq \|x - z\|^2$, $x \in H, z \in C$. We denote by $\text{Fix}(T) = \{x \in C : Tx = x\}$, the set of fixed points for a mapping T .

Lemma 1.1 ([16]) Let H be any real Hilbert space, let T_i be an α_i -averaged mapping with $\alpha_i > 0$ for each $i \in \{1, 2, \dots, S\}$ and let $\omega = (\omega_1, \omega_2, \dots, \omega_S)$ be a positive real vector such that $\sum_{i=1}^S \omega_i = 1$. Set $T = \sum_{i=1}^S \omega_i T_i$ and $\alpha = \sum_{i=1}^S \omega_i \alpha_i$. Then, T is α -averaged. Moreover, $\tilde{T} = T_S T_{S-1} \dots T_1$ is $\tilde{\alpha}$ -averaged with $\tilde{\alpha} = 1/(1 + 1/\sum_{i=1}^S \alpha_i/(1 - \alpha_i))$ and $\text{Fix}(T) = \text{Fix}(\tilde{T}) = \bigcap_{i=1}^S \text{Fix}(T_i)$.

Lemma 1.2 ([17]) Let C be a closed convex subset of a real Hilbert space H and let $T : C \rightarrow C$ be a nonexpansive mapping with $\text{Fix}(T) \neq \emptyset$. If $\{z^l\}$ is a sequence in C weakly converging to x and if $\{(E - T)z^l\}$ converges strongly to y , then $(E - T)x = y$. In particular, if $y = 0$, then $x \in \text{Fix}(T)$.

Lemma 1.3 ([18]) Let $\{a_k\}$ and $\{c_k\}$ be sequences of real numbers, having the properties: $a_{k+1} \leq (1 - t_k)a_k + t_k c_k$, $a_k \geq 0$, $\limsup_{k \rightarrow \infty} c_k \leq 0$ and t_k satisfies condition

(t): $t_k \in (0, 1)$ for all $k \geq 1$, $\lim_{k \rightarrow \infty} t_k = 0$ and $\sum_{k=1}^{\infty} t_k = \infty$.

Then, $\lim_{k \rightarrow \infty} a_k = 0$.

Lemma 1.4 ([19]) Let H be a real Hilbert space and let $F : H \rightarrow H$ be an η -strongly monotone and γ -strictly pseudocontractive mapping with $\eta + \gamma > 1$. Then, for any $t \in (0, 1)$, $E - tF$ is contractive with constant $1 - t\tau$ where $\tau = 1 - \sqrt{(1 - \eta)/\gamma}$.

Lemma 1.5 ([20]) Let $\{a_k\}$ be a sequence of real numbers with a subsequence $\{k_l\}$ of $\{k\}$ such that $a_{k_l} < a_{k_l+1}$ for all integer numbers $l \geq 1$. Then, there exists a nondecreasing sequence $\{m_k\} \subseteq \{k\}$ such that $m_k \rightarrow \infty$, $a_{m_k} \leq a_{m_k+1}$ and $a_k \leq a_{m_k+1}$ for all (sufficiently large) integer numbers $k \geq 1$. In fact, $m_k = \max\{l \leq k : a_l \leq a_{l+1}\}$.

2. Main results

2.1. String-averaged algorithmic scheme for the multiple-sets split feasibility problem

Let the string $I_t = (i_1^t, i_2^t, \dots, i_{\gamma(I_t)}^t)$ be a finite nonempty subset of I , for every $t = 1, 2, \dots, S_1$, where the length of the string I_t , denoted by $\gamma(I_t)$, is the number of elements in I_t . Put $T_t^1 := P_{i_{\gamma(I_t)}^t} \dots P_{i_2^t} P_{i_1^t}$, where $P_{i_l^t} = P_{C_{i_l^t}}$, for $l = 1, 2, \dots, \gamma(I_t)$ and $t = 1, 2, \dots, S_1$.

Given a positive weight vector $\beta = (\beta_1, \beta_2, \dots, \beta_{S_1})$ with $\sum_{t=1}^{S_1} \beta_t = 1$, we define the algorithmic mapping $\mathcal{P}_1 := \sum_{t=1}^{S_1} \beta_t T_t^1$. We suppose that every element of I appears in at least

one of the string I_t . Analogously, for the family $\{Q_j\}_{j \in J}$, we can construct the mapping $\mathcal{P}_2 := \sum_{t=1}^{S_2} \eta_t T_t^2$ where $T_t^2 := P_{j_t^1} \cdots P_{j_t^{S_1}} P_{j_t^1}$, $P_{j_t^1} = P_{Q_{j_t^1}}$ for $t = 1, 2, \dots, S_2, l = 1, 2, \dots, \gamma(I_t)$ and $\eta = (\eta_1, \eta_2, \dots, \eta_{S_2})$ is also a positive weight vector such that $\sum_{t=1}^{S_2} \eta_t = 1$.

First, we need to prove the following lemma.

Lemma 2.1 $z \in \Gamma$ if and only if $(E - \mathcal{P}_1)z = A^*(E - \mathcal{P}_2)Az = 0$. Moreover, the last equality holds if and only if $(E - \mathcal{P}_2)Az = 0$.

Proof. Clearly, when $z \in \Gamma$, we have that $z \in C_i$ and $Az \in Q_j$ for every $i \in I$ and $j \in J$. Consequently, for all $t = 1, 2, \dots, S_1$ we have that $T_t^1 z = z$ and for $t = 1, 2, \dots, S_2$, $T_t^2 Az = Az$. From the last two equalities and the properties of β_t and η_t it follows $(E - \mathcal{P}_1)z = A^*(E - \mathcal{P}_2)Az = 0$. Inversely, we have to prove that if z satisfies the equalities then $z \in \Gamma$. Take any point $p \in \Gamma$. It is easy to see that

$\|z - p\|^2 = \|\mathcal{P}_1 z - p\|^2 \leq \sum_{t=1}^{S_1} \beta_t \|T_t^1 z - p\|^2 \leq \|z - p\|^2 - \sum_{t=1}^{S_1} \beta_t \sum_{l=1}^{\gamma(I_t)} \|U^{i_t^l} z - U^{i_{l-1}^t} z\|^2$, where $U^{i_t^l} = P_{i_t^l} \cdots P_{i_2^t} P_{i_1^t}$ and $U^{i_0^t} = E$. Therefore, we get that $\|U^{i_t^l} z - U^{i_{l-1}^t} z\|^2 = 0$ for $l = 1, 2, \dots, \gamma(I_t)$, because $\beta_t > 0$. Taking $l = 1$, we obtain that $U^{i_1^t} z = z$, which together with the case that $l = 2$ implies $U^{i_2^t} z = z$. Repeating the process for $l = 3, \dots, \gamma(I_t)$, we get that $U^{i_l^t} z = z$ for $l = 3, \dots, \gamma(I_t)$. Finally, $U^{i_l^t} z = z$ for $l = 1, 2, \dots, \gamma(I_t)$ and $t = 1, 2, \dots, S_1$. Since each element of I appears in at least one I_t , $P_{C_i} z = z$ for each $i \in I$. So, in order to finish the proof for the first conclusion, it is sufficient to show that $P_{Q_j} Az = Az$ for all $j \in J$. Indeed, from $A^*(E - \mathcal{P}_2)Az = 0$ and the nonexpansivity of $\mathcal{P}_2$, we can write, respectively, that $\mathcal{P}_2 Az = Az - w$ where $A^* w = 0$ and

$\|Az - Ap\|^2 \geq \|\mathcal{P}_2 Az - \mathcal{P}_2 Ap\|^2 = \|Az - w - Ap\|^2 = \|Az - Ap\|^2 + \|w\|^2 - 2\langle A^* w, z - p \rangle$. Thus, $w = 0$. Consequently, $\mathcal{P}_2 Az = Az$. By the similar argument as the above, we get the first conclusion. For the second conclusion, we need only to prove the case that if $A^*(E - \mathcal{P}_2)Az = 0$ then $(E - \mathcal{P}_2)Az = 0$. Indeed, from the $(1/2)$ -inverse strongly monotone property of $(E - \mathcal{P}_2)$ [21], we get that

$$0 \leq (1/2) \|(E - \mathcal{P}_2)Az\|^2 \leq \langle (E - \mathcal{P}_2)Az - (E - \mathcal{P}_2)Ap, Az - Ap \rangle \\ = \langle A^*(E - \mathcal{P}_2)Az, z - p \rangle = \|A^*(E - \mathcal{P}_2)Az\| \|z - p\| = 0.$$

This completes the proof. $\square$

We have the following results.

Theorem 2.1 Let H_1 and H_2 be two real Hilbert spaces, let $A : H_1 \rightarrow H_2$ be a bounded linear mapping such that $A \neq 0$. Let C_i and Q_j , for each $i \in I$ and $j \in J$, be closed convex subsets in H_1 and H_2 , respectively. Assume that $\Gamma \neq \emptyset$. Then, the sequence $\{x^k\}$, defined by **Algorithmic scheme 1**:

$$x^{k+1} = Tx^k, \quad T = \mathcal{P}_1(E - \gamma A^*(E - \mathcal{P}_2)A), \quad (2.1)$$

where $\gamma \in (0, 1/\|A\|^2)$ is a fixed number, converges weakly to a solution of (1.1), as $k \rightarrow \infty$.

Proof. Since $P_{i_t^l}$, for $l = 1, 2, \dots, \gamma(I_t)$, is $(1/2)$ -averaged, by Lemma 1.1, the mapping T_t^1 is α_t -averaged with $\alpha_t = 1/(1 + 1/\gamma(I_t))$, and hence, $\mathcal{P}_1$ is a β -averaged mapping with $\beta = \sum_{t=1}^{S_1} \beta_t \alpha_t$. Meantime, since $\mathcal{P}_2$ is nonexpansive, by using Lemma 3.3 in [22], the mapping $E - \gamma A^*(E - \mathcal{P}_2)A$ is η -averaged, where $\eta = \gamma\|A\|^2$ and $0 < \gamma < 1/\|A\|^2$. Therefore, again by Lemma 1.1, the mapping T is τ -averaged with $\tau = (\beta + \eta - 2\beta\eta)/(1 - \beta\eta)$. Thus, from Lemmas 2.1 and 1.1, we have that $\text{Fix}(T) = \text{Fix}(\mathcal{P}_1) \cap \text{Fix}(E - \gamma A^*(E - \mathcal{P}_2)A) = \text{Fix}(\mathcal{P}_1) \cap \text{Zer}(A^*(E - \mathcal{P}_2)A) = \Gamma$, where $\text{Zer}(B)$ denotes the set of x such that $Bx = 0$ for any mapping B in H_1 . Further, by using Theorem 2.1 in [3], we get the conclusion. $\square$

Remark 2.1 Taking $S_1 = S_2 = 1$, we have that $\mathcal{P}_1 = P_{C_N} \cdots P_{C_2} P_{C_1}$ and $\mathcal{P}_2 = P_{Q_M} \cdots P_{Q_2} P_{Q_1}$.

When $S_1 = N$ and $S_2 = M$ with $\gamma(I_t) = \gamma(J_t) = 1$, we obtain that $\mathcal{P}_1 = \sum_{i=1}^N \beta_i P_{C_i}$ and $\mathcal{P}_2 = \sum_{j=1}^M \eta_j P_{Q_j}$. Then, the method (2.1) in this case contains (1.5), as particularities.

2.2. A general string-averaged algorithmic scheme for the multiple-sets split feasibility problem

In this section, in order to obtain the strong convergence, we propose a new algorithmic scheme based on the general iterative method for nonexpansive mappings, introduced by Marino and Xu [15].

Algorithmic scheme 2:

Step 1. Let x^1 and ε_1 be any point in H_1 and any positive real number, respectively. Set $k := 1$;
 Step 2. Assume that the k th iterate x^k has been constructed. If $(E - \mathcal{P}_1)x^k = (E - \mathcal{P}_2)Ax^k = 0$ then stop and x^k is a solution of (1.1). Otherwise, compute

$$u^k = \mathcal{P}_1(E - \gamma_k A^*(E - \mathcal{P}_2)A)x^k, \quad x^{k+1} = t_k \bar{\gamma} f x^k + (E - t_k F)u^k, \quad (2.2)$$

where $\gamma_k = \rho_k q(x^k) / \|A^*(E - \mathcal{P}_2)Ax^k\|^2$ if $(E - \mathcal{P}_2)Ax^k \neq 0$ and

$$\gamma_k = \frac{\rho_k q(x^k)}{(\|A^*(E - \mathcal{P}_2)Ax^k\| + \varepsilon_k)^2}, \quad (2.3)$$

if $(E - \mathcal{P}_2)Ax^k = 0$, with $q(x) = (1/2) \sum_{t=1}^{S_2} \eta_t \|(E - T_t^2)Ax\|^2$, a new proximity function, $f : H_1 \rightarrow H_1$ is a θ -Lipschitz continuous mapping with a Lipschitz constant $\theta \in [0, \infty)$ and $F : H_1 \rightarrow H_1$ is an η -strongly monotone and γ -strictly pseudocontractive mapping.

Step 3. Set $k := k + 1$ and go to Step 2.

We assume parameters ρ_k and ε_k for all $k \geq 1$, satisfy, respectively, the following conditions:

(ρ): $0 < \underline{\rho} \leq \rho_k \leq \bar{\rho} < 2$ and

(ε): $\{\varepsilon_k\}$ is a bounded sequence of positive real numbers such that $\liminf_{k \rightarrow \infty} \varepsilon_k > 0$.

For the sake of simplicity, when $(E - \mathcal{P}_1)x^k \neq 0$, the next iterate x^{k+1} can be calculated by (2.2) and (2.3) without verifying that $(E - \mathcal{P}_2)Ax^k$ equal or does not equal to zero.

Theorem 2.2 Let H_1, H_2, A, C_i and Q_j with Γ be as in Theorem 2.1. Let $f : H_1 \rightarrow H_1$ be a θ -Lipschitz continuous mapping with a Lipschitz constant $\theta \in [0, \infty)$ and let $F : H_1 \rightarrow H_1$ be an η -strongly monotone and γ -strictly pseudocontractive mapping such that $\eta + \gamma > 1$. Assume that there hold conditions (ρ), (ε) and (t), $\bar{\gamma} \in (0, \tau/\theta)$ is a fixed number with $\tau = 1 - \sqrt{(1 - \eta)/\gamma}$ when $\theta \neq 0$ and $\bar{\gamma} = 0$ when $\theta = 0$. Then, the sequence $\{x^k\}$, defined by Algorithmic scheme 2, converges strongly to a point in Γ , as $k \rightarrow \infty$.

Proof. We only discuss the case when the algorithm does not stop after finite number of iterates.

First, we consider the case when $\theta \neq 0$. We prove that $\{x^k\}$, defined by Algorithmic scheme 2, is bounded. Take a point $p \in \Gamma$. Then, since $\mathcal{P}_1$ and T_t^2 are nonexpansive and $E - T_t^2$ is $(1/2)$ -inverse strongly monotone [21] for each $t = 1, 2, \dots, S_2$, we have that

$$\begin{aligned} \|u^k - p\|^2 &= \|\mathcal{P}_1(E - \gamma_k A^*(E - \mathcal{P}_2)A)x^k - \mathcal{P}_1 p\|^2 \leq \|x^k - p - \gamma_k A^*(E - \mathcal{P}_2)Ax^k\|^2 \\ &= \|x^k - p\|^2 - 2\gamma_k \langle (E - \mathcal{P}_2)Ax^k - (E - \mathcal{P}_2)Ap, Ax^k - Ap \rangle \\ &\quad + \gamma_k^2 \|A^*(E - \mathcal{P}_2)Ax^k\|^2 \\ &= \|x^k - p\|^2 - 2\gamma_k \sum_{t=1}^{S_2} \eta_t \langle (E - T_t^2)Ax^k - (E - T_t^2)Ap, Ax^k - Ap \rangle \\ &\quad + \gamma_k^2 \|A^*(E - \mathcal{P}_2)Ax^k\|^2 \\ &\leq \|x^k - p\|^2 - 2\gamma_k q(x^k) + \gamma_k^2 (\|A^*(E - \mathcal{P}_2)Ax^k\| + \varepsilon_k)^2 \\ &= \|x^k - p\|^2 - \rho_k(2 - \rho_k) \frac{q^2(x^k)}{(\|A^*(E - \mathcal{P}_2)Ax^k\| + \varepsilon_k)^2} \leq \|x^k - p\|^2. \end{aligned} \quad (2.4)$$

Therefore,

$$\begin{aligned}\|x^{k+1} - p\| &= \|t_k \bar{\gamma}(fx^k - fp) + (E - t_k F)u^k - (E - t_k F)p - t_k Fp + t_k \bar{\gamma}fp\| \\ &\leq t_k \bar{\gamma} \theta \|x^k - p\| + (1 - t_k \tau) \|u^k - p\| + t_k \|Fp - \bar{\gamma}fp\| \\ &\leq (1 - t_k(\tau - \bar{\gamma}\theta)) \|x^k - p\| + t_k \|Fp - \bar{\gamma}fp\| \\ &\leq \max\{\|x^1 - p\|, \|Fp - \bar{\gamma}fp\|/(\tau - \bar{\gamma}\theta)\},\end{aligned}$$

from which it follows the boundedness of $\{x^k\}$. Further, we have the following estimate

$$\begin{aligned}\|x^{k+1} - p\|^2 &\leq (1 - t_k(\tau - \bar{\gamma}\theta)) \|x^k - p\|^2 - \tilde{\rho} \frac{q^2(x^k)}{(\|A^*(E - \mathcal{P}_2)Ax^k\| + \varepsilon_k)^2} \\ &\quad - 2t_k \langle Fp - \bar{\gamma}fp, x^{k+1} - p \rangle,\end{aligned}\tag{2.5}$$

where $\tilde{\rho}$ is some positive constant such that $(1 - t_k \tau) \rho_k (2 - \rho_k) \geq \tilde{\rho}$ for all $k \geq 1$. We need only to consider two cases.

Case 1. $\|x^{k+1} - p\| \leq \|x^k - p\|$ for all $k \geq k_0$ large enough. Then, there exists $\lim_{k \rightarrow \infty} \|x^k - p\|$. From (2.5) it follows that

$$\begin{aligned}0 &\leq \frac{\tilde{\rho} q^2(x^k)}{(\|A^*(E - \mathcal{P}_2)Ax^k\| + \varepsilon_k)^2} \leq \|x^k - p\|^2 - \|x^{k+1} - p\|^2 \\ &\quad - t_k(\tau - \bar{\gamma}\theta) \|x^k - p\|^2 + 2t_k \|Fp - \bar{\gamma}fp\| \|x^{k+1} - p\|.\end{aligned}\tag{2.6}$$

So, from (2.6), the existence of $\lim_{k \rightarrow \infty} \|x^k - p\|$ with the boundedness of $\{x^k\}$ and property of $\{\varepsilon_k\}$, we get that $\lim_{k \rightarrow \infty} q(x^k) = 0$. Thus, $\lim_{k \rightarrow \infty} \|(E - T_t^2)Ax^k\| = 0$ for all $t = 1, \dots, S_2$, and hence,

$$\lim_{k \rightarrow \infty} \|(E - \mathcal{P}_2)Ax^k\| = 0\tag{2.7}$$

and $\lim_{k \rightarrow \infty} \|A^*(E - \mathcal{P}_2)Ax^k\| = 0$. Put $v^k := -\gamma_k A^*(E - \mathcal{P}_2)Ax^k$ and $z^k = x^k + v^k$. It is clear that $\lim_{k \rightarrow \infty} v^k = 0$ and

$$\begin{aligned}\|u^k - p\|^2 &= \|\mathcal{P}_1 z^k - p\|^2 \leq \sum_{t=1}^{S_1} \beta_t \|T_t^1 z^k - p\|^2 \\ &\leq \|z^k - p\|^2 - \sum_{t=1}^{S_1} \beta_t \sum_{l=1}^{\gamma(I_t)} \|U^{i_l^t} z^k - U^{i_{l-1}^t} z^k\|^2 \\ &\leq \|x^k - p\|^2 + 2\|v^k\| \|z^k - p\| - \sum_{t=1}^{S_1} \beta_t \sum_{l=1}^{\gamma(I_t)} \|U^{i_l^t} z^k - U^{i_{l-1}^t} z^k\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\|x^{k+1} - p\|^2 &\leq t_k \bar{\gamma} \theta \|x^k - p\|^2 + \|x^k - p\|^2 + 2\|v^k\| \|z^k - p\| \\ &\quad - \sum_{t=1}^{S_1} \beta_t \sum_{l=1}^{\gamma(I_t)} \|U^{i_l^t} z^k - U^{i_{l-1}^t} z^k\|^2 + 2t_k \|Fp - \bar{\gamma}fp\| \|x^{k+1} - p\|,\end{aligned}$$

and hence, $\lim_{k \rightarrow \infty} \|U^{i_l^t} z^k - U^{i_{l-1}^t} z^k\| = 0$ for all $l = 1, 2, \dots, \gamma(I_t)$ and all $t = 1, 2, \dots, S_1$. As in the proof of Lemma 2.1,

$$\lim_{k \rightarrow \infty} \|(E - P_{C_i})x^k\| = 0 \quad \forall i \in I.\tag{2.8}$$

Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_m}\}$ of the sequence $\{x^k\}$ such that x^{k_m} converges weakly to a point $\tilde{p} \in H_1$ as $m \rightarrow \infty$. Then, Ax^{k_m} also converges weakly to $A\tilde{p}$. From Lemmas 1.2 and 2.1 with (2.7) and (2.8) it follows that $\tilde{p} \in \Gamma$. Similarly, we have also that every cluster point of $\{x^k\}$ belongs to Γ . Therefore,

$$\limsup_{k \rightarrow \infty} \langle Fp_* - \bar{\gamma}fp_*, p_* - x^k \rangle = \lim_{m \rightarrow \infty} \langle Fp_* - \bar{\gamma}fp_*, p_* - x^{k_m} \rangle = \langle Fp_* - \bar{\gamma}fp_*, p_* - \tilde{p} \rangle \leq 0,$$

where p_* is a unique solution of $p_* \in \Gamma : \langle Fp_* - \bar{\gamma}fp_*, p_* - p \rangle \leq 0 \quad \forall p \in \Gamma$. Now, from (2.5) we have that

$$\|x^{k+1} - p_*\|^2 \leq (1 - t_k(\tau - \bar{\gamma}\theta)) \|x^k - p_*\|^2 + 2t_k \langle Fp_* - \bar{\gamma}fp_*, p_* - x^{k+1} \rangle.$$

By Lemma 1.3, $\|x^k - p_*\| \rightarrow 0$ as $k \rightarrow \infty$.

Case 2. There exists a subsequence $\{k_l\}$ of $\{k\}$ such that $\|x^{k_l} - p\| < \|x^{k_l+1} - p\|$ for all $l \geq 0$. Hence, by Lemma 1.5, there exists a nondecreasing sequence $\{m_k\} \subseteq \{k\}$ such that $m_k \rightarrow \infty$,

$$\|x^{m_k} - p\| \leq \|x^{m_k+1} - p\| \quad \text{and} \quad \|x^k - p\| \leq \|x^{m_k+1} - p\| \quad (2.9)$$

for each $k \geq 1$. Then, from (2.5) and the first inequality in (2.9), we know that

$$\|x^{m_k} - p\|^2 \leq (2/(\tau - \bar{\gamma}\theta)) \langle Fp - \bar{\gamma}fp, p - x^{m_k+1} \rangle. \quad (2.10)$$

In this case, instead of (2.6), we get that

$$0 \leq \frac{\tilde{p}q^2(x^{m_k})}{\|A^*(E - \mathcal{P}_2)Ax^{m_k}\| + \varepsilon_{m_k}} \leq 2t_{m_k} \|Fp - \bar{\gamma}fp\| \|x^{m_k+1} - p\|,$$

and hence, $\lim_{k \rightarrow \infty} q(x^{m_k}) = 0$. By the similar argument as in the proof for the case 1, $\lim_{k \rightarrow \infty} \|(E - \mathcal{P}_2)Ax^{m_k}\| = 0$ and $\lim_{k \rightarrow \infty} \|(E - P_{C_i})x^{m_k}\| = 0$ for all $i \in I$ and any cluster point of $\{x^{m_k}\}$ belongs to Γ . Thus, $\limsup_{k \rightarrow \infty} \langle Fp_* - \bar{\gamma}fp_*, p_* - x^{m_k+1} \rangle \leq 0$, which together with (2.10) implies that $\|x^{m_k} - p_*\| \rightarrow 0$ as $k \rightarrow \infty$. Now, from (2.5) with k and p replaced, respectively, by m_k and p_* it follows that $\|x^{m_k+1} - p_*\| \rightarrow 0$. Noting the second inequality in (2.9), $\|x^k - p_*\| \rightarrow 0$.

In the case that $\theta = 0, \bar{\gamma} = 0$. Hence, (2.2) has the expression

$$x^{k+1} = (E - t_k F) \mathcal{P}_1 (E - \gamma_k A^* (E - \mathcal{P}_2) A) x^k. \quad (2.11)$$

Replacing $\bar{\gamma} = 0$ in the proof when $\bar{\gamma} \neq 0$, we obtain the conclusion. The proof is completed. $\square$

Remark 2.2 It is easy to see that $F = E - f$ is also η -strongly monotone and γ -strictly pseudocontractive such that $\eta + \gamma > 1$, where $f = aE + (1 - a)u$ with a fixed $u \in H_1$ and $a \in (0, 1)$. Then, replacing $F = E - f$ in (2.11), we obtain a modified Halpern's algorithmic scheme,

$$x^{k+1} = t_k u + (1 - t_k) \mathcal{P}_1 (E - \gamma_k A^* (E - \mathcal{P}_2) A) x^k, \quad (2.12)$$

with a new $t_k := (1 - a)t_k$.

From Theorem 2.2, we have the following result for the strong convergence of Halpern's algorithmic scheme (2.12) with (2.3).

Theorem 2.3 Let H_1, H_2, A, C_i and Q_j with Γ be as in Theorem 2.1. Assume that there hold conditions (ρ) , (ε) and (t) . Then, the sequence $\{x^k\}$, defined by the algorithmic scheme (2.12) with γ_k defined in Algorithmic scheme 2, converges strongly to a point in Γ , as $k \rightarrow \infty$.

3. Conclusion

In this paper, to solve the multiple-sets split feasibility problem (MSSFP) in Hilbert spaces, we proposed two string-averaged algorithmic schemes, one of which was given with the strong convergence and a self-adaptive step-size. We also considered the particular case of the proposed schemes.

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