INVERSE MOMENTS FOR GENERALIZATION OF FRACTIONAL BESSEL TYPE PROCESS

Vu Thi Huong
University of Transport and Communications

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ABSTRACT

This paper considers a generalization of fractional Bessel type process. It is also a type of singular stochastic differential equations driven by fractional Brownian motion which has been studied by some authors. The purpose of this paper is to study inverse moments problem for this type of equation. We applied the techniques of Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion. We obtain that under some assumptions of coefficients, the inverse moments of solution are bounded. This result is useful to estimate the rate of convergence of the numerical approximation in the *Lp*- norm.

MOMENT NGƯỢC CỦA QUÁ TRÌNH BESSEL P HÂ N THỨ DẠ NG TỔNG QUÁT

Vũ Thị Hương
Trường Đại học Giao thông Vận tải

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TỪ KHÓA

Chuyển động Brown phân thứ
Quá trình Bessel phân thứ
Phương trình vi phân ngẫu nhiên phân thứ
Tính toán Malliavin
Moment ngược

TÓM TẮT

Bài báo này xem xét một dạng tổng quát của quá trình Bessel phân thứ. Đây cũng là một dạng thuộc lớp các phương trình vi phân ngẫu nhiên kỳ dị xác định bởi chuyển động Brown phân thứ đã được nghiên cứu bởi một số tác giả. Mục đích chính của bài báo là nghiên cứu moment ngược của quá trình này. Chúng ta sử dụng tính toán Malliavin cho phương trình vi phân ngẫu nhiên xác định bởi chuyển động Brown phân thứ. Với một số giả thiết của các hệ số, chúng ta đánh giá được tính bị chặn của moment ngược. Đây là một đánh giá cần thiết khi xem xét tốc độ hội tụ của nghiệm xấp xỉ trong *Lp*.

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Email: vthuong@utc.edu.vn

1. Introduction

In [1], author considered a more general singular stochastic differential equation driven by fractional Brownian motion. More precisely, we study a generalization of the Bessel type process $Y = (Y(t))_{0\le t\le T}$ satisfying the following SDEs,

$$dY(t) = \left(\frac{k}{Y(t)} + b(t, Y(t))\right) dt + \sigma dB^H(t), \tag{1}$$

where $0 \le t \le T$, $Y(0) > 0$ and B^H is a fractional Brownian motion with the Hurst parameter $H > \frac{1}{2}$ defined in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t, t \in [0,T]\}$ satisfying the usual condition. Fix $T > 0$ and we consider equation (1) on the interval $[0,T]$. We suppose that $k > 0$ and the coefficient $b = b(t,x) : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ are mesurable functions and globally Lipschitz continuous with respect to x, linearly growth with respect to x. It means that there exists positive constants L, C such that the following conditions hold:

A_1) $|b(t,x) - b(t,y)| \le L|x-y|$, for all $x, y \in \mathbb{R}$ and $t \in [0,T]$;

A_2) $|b(t,x)| \le C(1+|x|)$, for all $x \in \mathbb{R}$ and $t \in [0,T]$.

In [1], author proved that under some assumptions of cofficients, this equation has a unique positive solution. Moreover, in [2], author showed that the Malliavin derivative for this process is an exponent function of the drift coefficient's derivative. In this paper, we estimate the inverse moments of the solution using the Malliavin calculus for stochastic differential equations driven by a fractional Brownian motion. This is an interesting problem that has been studied by some authors because it is necessary in showing the rate convegence of the numerical approximation in the L_p - norm. We can see some results in [3-5]. Firstly, we shall recall some basic facts on Malliavin calculus (see [6-8]).

2. Malliavin calculus

Fix a time interval $[0,T]$. We consider a fractional Brownian motion $\{B^H(t)\}_{t\in[0,T]}$ We note that $E(B^H(s).B^H(t)) = R_H(s,t)$ where

$$R_H(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

We denote by $\mathcal{E}$ the set of step functions on $[0,T]$ with values in $\mathbb{R}^m$. Let $\mathcal{H}$ be the Hibert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]}\rangle_{\mathcal{H}} = R_H(t,s).$$

On the other hand, the covariance $R_H(t,s)$ can be written as

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |r-u|^{2H-2} du dr = \int_0^{t\wedge s} K_H(t,r) K_H(s,r) dr,$$

where $\alpha_H = H(2H-1), K_H(t,s)$ is the square integrable kernel defined by

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_0^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

for $t > s$, where $c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$ and β denote the Beta function. We put $K_H(t,s) = 0$ if $t \le s$.

It implies that for all $\varphi, \psi \in \mathcal{H}$

$$\langle \varphi, \psi \rangle = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi_r \psi_u du dr \tag{2}$$

The mapping $1_{[0,t]} \mapsto B^H(t)$ can be extend to an isometry between $\mathcal{H}$ and the Gaussian space associated with B^H. Denote this isometry by $\varphi \mapsto B(\varphi)$.

Let $\mathcal{S}$ be the space of smooth and cylindrical random variables of the form

$$F = f(B^H(\varphi_1), \ldots, B^H(\varphi_n)),$$

where $n \ge 1, f \in C_b^\infty(\mathbb{R}^n)$. We define the derivative operator DF on $F \in \mathcal{S}$ as the $\mathcal{H}$ -valued random variable

$$DF = \sum_{i=1}^n \frac{\partial F}{\partial x_i}(B^H(\varphi_1), \ldots, B^H(\varphi_n))\varphi_i.$$

We denote by $\mathbb{D}^{1,2}$ the Sobolev space defined as the completion of the class $\mathcal{S}$, with respect to the norm

$$\|F\|_{1,2} = \left[\mathbb{E}(F^2) + \mathbb{E}(\|DF\|^2_{\mathcal{H}}\right]^{1/2}.$$

We denote by δ the adjoint of the derivative operator D. We say $u \in Dom\delta$ if there is a $\delta(u) \in L^2(\Omega)$ such that for any $F \in \mathbb{D}^{1,2}$ the following duality relationship holds:

$$E(\langle u, DF\rangle_{\mathcal{H}}) = E(\delta(u)F).$$

The random variable $\delta(u)$ is also called the Skorohod integral of u with respect to the fBm B_j, and we use the notation $\delta(u) = \int_0^T u(t)\delta B^H(t)$.

Suppose that $u = \{u(t), t \in [0,T]\}$ is a stochastic process whose trajectories are Holder continuous of order $\gamma > 1 - H$. Then, the Riemann–Stieltjes integral $\int_0^t u(t)dB^H(t)$ exists. On the other hand, if $u \in \mathbb{D}^{1,2}(\mathcal{H})$ and the derivative $D_s^j u(t)$ exists and satisfies almost surely

$$\int_0^T \int_0^T |D_s^j u(t)||t-s|^{2H-2} ds dt < \infty,$$

and $\mathbb{E}(\|Du\|^2_{L^{1/H}([0,T]^2)}) < \infty$, then (see Proposition 5.2.3 in [7]) $\int_0^T u(t)\delta B^H(t)$ exists, and we have the following relationship between these two stochastic integrals.

Lemma 2.1.

$$\int_0^T u(t)dB^H(t) = \int_0^T u(t)\delta B^H(t) + \alpha_H \int_0^T \int_0^T D_s u(t)|t-s|^{2H-2} ds dt, \tag{3}$$

where $\alpha_H = H(2H-1)$.

Following paper [2] we can estimate the Malliavin derivative of $Y(t)$.

Lemma 2.2. *Assume that conditions* $(A1)-(A2)$ *are satisfied and* $Y(t)$ *is the solution of equation* (1), *then for any* $t>0$, *we have*

$$DY(t) = \sigma.exp\left(\int_.^t \left[\frac{k}{Y^2(r)} + \frac{\partial b}{\partial y}(r, Y(r))\right] dr\right) \mathbf{1}_{[0,t]}. \tag{4}$$

3. Inverse moments of generalization of fractional Bessel type process

The following theorem consider the negative moments for the solution of the equation (1). This is the main result of this paper. It states that

Theorem 3.1. *Assume that conditions* $(A1)-(A2)$ *are satisfied and* $Y(t)$ *is the solution of the equation* (1). *For* $p \geq 1$ *with*

$$\frac{k}{2} \geq (p+1)H\sigma^2 s^{2H-1} e^{Ls}, s \in [0,T],$$

then

$$\sup_{t\in[0,T]} \mathbb{E}[Y(t)]^{-p} < \infty.$$

Proof. Applying chain rule for Riemann–Stieltjes integral we have

$$\begin{aligned}
(Y(t)+\epsilon)^{-p} = &(Y(0)+\epsilon)^{-p} - p\int_0^t \frac{f(s,Y(s))}{(\epsilon+Y(s))^{p+1}} ds - p\int_0^t \sigma \frac{1}{(\epsilon+Y(s))^{p+1}} dB^H(s) \\
&\leq (Y_i(0)+\epsilon)^{-p} - p\int_0^t \frac{f(s,Y(s))}{(\epsilon+Y(s))^{p+1}} ds - p\int_0^t \sigma \frac{1}{(\epsilon+Y(s))^{p+1}} \delta B^H(s) \\
&- \alpha_H p\sigma \int_0^t \int_0^t D_r\left(\frac{1}{(\epsilon+Y(s))^{p+1}}\right) |s-r|^{2H-2} ds dr.
\end{aligned}$$

We have

$$\begin{aligned}
&-\alpha_H p\sigma \int_0^t \int_0^t D_r\left(\frac{1}{(\epsilon+Y(s))^{p+1}}\right)|s-r|^{2H-2} ds dr \\
&= \alpha_H p(p+1)\sigma \int_0^t \int_0^t \frac{D_r Y(s)}{(\epsilon+Y(s))^{p+2}} |s-r|^{2H-2} ds dr
\end{aligned}$$

$$= p(p+1)Hs^{2H-1}\sigma \int_0^t \frac{D_r Y(s)}{(\epsilon + Y(s))^{p+2}} dr.$$

By applying Lemma (2.2)

$$\begin{aligned} &- \alpha_H p\sigma \int_0^t \int_0^t D_r \left(\frac{1}{(\epsilon + Y(s))^{p+1}} \right) |s-r|^{2H-2} ds dr \\ &= p(p+1)Hs^{2H-1} \int_0^t \sigma^2 \frac{exp\left(\int_r^t \partial_y f(u, Y(u)du\right) \mathbf{1}_{[0,s]}(r)}{(\epsilon + Y(s))^{p+2}} dr. \end{aligned}$$

But

$$\begin{aligned} &(f(s,x) - f(s,y))(x-y) = \left(\frac{k}{x} - \frac{k}{y}\right)(x-y) + (b(s,x) - b_i(s,y))(x-y) \\ &\le (b(s,x) - b(s,y))(x-y) \le L|x-y|^2, \ for\ all x, y \in (0, +\infty). \end{aligned}$$

It implies that $\dfrac{f(s,x) - f(s,y)}{x-y} \le L$. *It mean that* $\partial_y f(s,y) < L$. *So we have*

$$\begin{aligned} &- \alpha_H p\sigma \int_0^t \int_0^t D_r \left(\frac{1}{(\epsilon + Y(s))^{p+1}} \right) |s-r|^{2H-2} ds dr \\ &\le p(p+1)Hs^{2H-1}\sigma^2 \int_0^t \frac{e^{\int_0^s L du}}{(\epsilon + Y(s))^{p+2}} dr. \end{aligned}$$

Then

$$\begin{aligned} (Y(t)+\epsilon)^{-p} \le (Y(0)+\epsilon)^{-p} - p\int_0^t \frac{f(s, Y(s)Y(s) - (p+1)Hs^{2H-1}\sigma^2 e^{Ls}}{(\epsilon + Y(s))^{p+2}} ds \\ - p\int_0^t \sigma \frac{1}{(\epsilon + Y(s))^{p+1}} \delta B^H(s). \end{aligned}$$

Moreover, the function $f(t,y)$ *satisfies the following properties*

(i) $f(t,y) \ge \dfrac{k}{2} y^(- 1)$ *for all* $y \le y_1 = \dfrac{\sqrt{C^2 + 2kC} - C}{2C}$.

(ii) $[f(t,y)]^- \le 2C(1+y^2), \forall y > 0, t > 0$ *where* $[f(t,y)]^-$ *is the negative parts of the function* $f(t,y)$.

Then

$$-\frac{f(t,y)}{(\epsilon+y)^{p+2}} \le -\mathbf{1}_{\{y \le y_1\}} \frac{k}{2y(\epsilon+y)^{p+2}} + \mathbf{1}_{\{y \ge y_1\}} \frac{2C(1+y^2)}{\epsilon + y)^{p+2}} \le 2C(\frac{1}{y_1^{p+2}} + \frac{1}{y_1}).$$

And

$$- \frac{f(s,y)y + (p+1)Hs^{2H-1}\sigma^2 e^{Ls}}{(\epsilon+y)^{p+2}}$$

$$\leq \frac{-\frac{k}{2}+(p+1)Hs^{2H-1}\sigma^2 e^{Ls}}{(\epsilon+y)^{p+2}}\mathbf{1}_{y\leq y_1}+\frac{2C(1+y^2)y+(p+1)Hs^{2H-1}\sigma^2 e^{Ls}}{(\epsilon+y)^{p+2}}\mathbf{1}_{y\geq y_1}$$
$$\leq \frac{-\frac{k}{2}+(p+1)Hs^{2H-1}\sigma^2 e^{Ls}}{(\epsilon+y)^{p+2}}\mathbf{1}_{y\leq y_1}+(p+1)Hs^{2H-1}\sigma^2 e^{Ls}y_1^{-p-2}+2C(y_1^{-p-1}+y_1^{-p+1}).$$

For $p\geq 1$ *and* $\frac{k}{2}\geq (p+1)Hs^{2H-1}\sigma^2 e^{Ls}$, *there exists the constant* $C_{y_1,p,\sigma}$ *such that*

$$(Y(t)+\epsilon)^{-p}\leq (Y(0)+\epsilon)^{-p}+C_{y_1,p,\sigma}\int_0^t (2C+s^{2H-1})ds-p\int_0^t \sigma\frac{1}{(\epsilon+Y(s))^{p+1}}\delta B^H(s).$$

Take expectation and letting $\epsilon\to 0$, *we obtain the conclusion.*

4. Conclusion

The main result of this paper is to estimate inverse moments for a generalization of fractional Bessel type process which is necessary in showing the rate convegence of the numerical approximation in the L_p - norm.

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