A NEW ITERATIVE METHOD FOR SOLVING PSEUDOMONOTONE VARIATIONAL INEQUALITIES

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ABSTRACT

In this paper, we introduce a modified algorithm for pseudomono- tone variational inequalities. This problem has many important applications in different fields such as optimization problem, Nash equilibrium problem, game theory, traffic equilibrium problem, fixed point problem. The proposed algorithm bases on the self-adaptive method and the modified Popov extragradient method that have been applied to solve many other problems with Lipschitz continuous mapping. The advantage of the algorithm is that it only needs to compute one value of the inequality mapping as well as it does not require knowing the Lipschitz constants of the variational inequality mapping. Moreover, our algorithm does not require its step-sizes tending to zero. This feature helps to speed up our method. The convergence of the method has been proved based on the specified conditions of the parameters. A numerical experiment in Euclidean spaces is given to illustrate the convergence of the new algorithm.

MỘT PHƯƠNG PHÁP LẶP MỚI GIẢI BẤT ĐẲNG THỨC BIẾN PHÂN GIẢ ĐƠN ĐIỆU

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TỪ KHÓA

Bất đẳng thức biến phân
Liên tục Lipschitz
Giả đơn điệu
Thuật toán đạo hàm tăng cường
Thuật toán tự thích nghi

TÓM TẮT

Trong bài báo này, chúng tôi giới thiệu một thuật toán cải tiến để giải các bài toán bất đẳng thức biến phân giả đơn điệu. Bài toán có nhiều ứng dụng quan trọng trong các lĩnh vực khác nhau như bài toán tối ưu, bài toán cân bằng Nash, lý thuyết trò chơi, bài toán cân bằng giao thông, bài toán điểm bất động. Thuật toán đề xuất đưa ra dựa trên phương pháp tự thích nghi và phương pháp đạo hàm tăng cường Popov đã được áp dụng để giải các bài toán bất đẳng thức biến phân với ánh xạ giá liên tục Lipschitz. Ưu điểm của thuật toán là chỉ cần tính toán một giá trị của ánh xạ bất đẳng thức và thuật toán không yêu cầu biết trước hệ số Lipschitz của ánh xạ bất đẳng thức biến phân. Ngoài ra, thuật toán của chúng tôi không yêu cầu bước nhảy tiến đến 0. Tính chất này giúp tăng tốc độ của thuật toán. Sự hội tụ của thuật toán đã được chứng minh dựa trên các điều kiện xác định của các tham số. Một ví dụ số được đưa ra để minh họa cho sự hội tụ của thuật toán mới.

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1. Introduction

Let C be a nonempty, closed and convex set in Euclidean space $\mathbb{R}^M$, $A : C \to \mathbb{R}^M$ be a mapping. The variational inequality problem of A on C is

$$\text{find } x^* \in C \text{ such that } \langle A(x^*), y - x^* \rangle \geq 0 \; \forall y \in C. \tag{VI(A,C)}$$

This problem has many important applications in economics, operations research, and mathematical physics [1] - [3]. It includes many problems of nonlinear analysis in a unified form, such as optimization, Nash equilibrium problems, game theory. Many authors have studied several algorithms to solve the problem VI(A,C). The simple one is the extragradient method [4] proposed by Korpelevich in a finite dimensional Euclidean space $\mathbb{R}^M$:

$$\begin{cases} x^0 \in C, \\ y^{n+1} = P_C(x^n - \lambda A(x^n)), \\ x^{n+1} = P_C(x^n - \lambda A(y^{n+1})). \end{cases} \tag{1}$$

Then A is psedomonotone and L-Lipschitz continuous on C, $\lambda \in (0, \frac{1}{L})$, the sequence $\{x^n\}$ generated by (1) converges to a solution of VI(A,C). This algorithm has been investigated and developed by a lot of authors [5] - [6]. The limitation of Korpelevich's method is that it computes the values of A mapping at two different points and requires knowing the Lipschitz constant L.

To deal with the case when the Lipschitz constant L is unknown, Yang and Liu [7] proposed the following adaptive stepsize strategy for:

$$\begin{cases} x^0 \in C, \\ y^n = P_C(x^n - \lambda_n A(x^n)), \\ x^{n+1} = y^n + \lambda_n (A(x^n) - A(y^n)), \end{cases} \tag{2}$$

where

$$\lambda_{n+1} = \begin{cases} \min\left\{ \frac{\rho \|x^n - y^n\|}{A(x^n) - A(y^n)}, \alpha_n \right\} & \text{if } A(x^n) \neq A(y^n), \\ \alpha_n & \text{if otherwise,} \end{cases}$$

$\rho \in (0,1)$ and $\lambda_0 > 0$. However, compared to (1), the Yang's algorithm requires the mapping A being psedomonotone and L-Lipschitz continuous on the whole space instead of the feasible set.

The second limitation can affect the applicability and efficiency of this method, which is not present in the Popov extragradient algorithm [8] proposed as a modification of the Arrow–Hurwicz algorithm. For solving the problem VI(A,C), Popov's extragradient algorithm $\{x^n\}$ is suggested as follows:

$$\begin{cases} x^0, y^0 \in C, \\ x^{n+1} = P_C(x^n - \lambda A(y^n)), \\ y^{n+1} = P_C(x^{n+1} - \lambda A(y^n)), \end{cases} \tag{3}$$

where $\lambda \in (0, \frac{1}{3L})$. In [8], the convergence of the sequences $\{x^n\}$ and $\{y^n\}$ generated by this method is proved.

The modified Popov algorithm [8] has been recently interested by many authors. They have improved and extended this method in different ways to obtain the weak and strong convergence of the method [6], [9].

Motivated by the works in [10], in this paper, we propose a new algorithm for solving VI(A,C). The new algorithm is improved from the Popov's algorithm and Hai's self-adaptive extragradient algorithm. Our method preserves the advantages and overcomes the drawbacks of the existing ones.

The new algorithm uses dynamic step sizes: the value of λ_n is selected based on the information of the previous steps, hence, it does not require to know the Lipschitz constant of A. On the other hand, the step-sizes in our algorithm are bounded away from zero. Moreover, at each iteration of the algorithm, we only need to compute the value of the mapping A at a point. A numerical experiment shows that these modifications improve the performance of the new algorithm.

The remaining part of this paper is organized as follows: the next section presents some definitions and lemmas that will be used for proving the convergence of the algorithm. The third section is devoted to the proof of our convergence results and a numerical example for illustration of the convergence of the new method.

2. Preliminaries

This section presents the definitions and properties of the projection operator; some definitions of the monotone, pseudomonotone and L-Lipschitz continuous mapping in $\mathbb{R}^M$; some results which will be used for proving the convergence of the algorithm. We refer the reader to [11] - [13] for more details.

For each $x \in \mathbb{R}^M$, denote

$$P_C(x) := \operatorname{argmin}\{\|z-x\| : z \in C\}.$$

The mapping $P_C : x \mapsto P_C(x)$ is called the metric projection from $\mathbb{R}^M$ onto C.

Proposition 2.1. [11] *It holds that:*

(i) $\|P_C(x) - P_C(y)\| \leq \|x-y\|$ *for all* $x, y \in \mathbb{R}^M$,

(ii) $\langle y - P_C(x), x - P_C(x)\rangle \leq 0$ *for all* $x \in \mathbb{R}^M, y \in C$.

Definition 2.1. [12] A mapping $A : C \to \mathbb{R}^M$ is called

1. monotone on C if for all $x, y \in C$,

$$\langle A(x) - A(y), x - y\rangle \geq 0.$$

2. pseudomonotone on C if for all $x, y \in C$, we have

$$\langle A(y), x - y\rangle \geq 0 \Rightarrow \langle A(x), x - y\rangle \geq 0.$$

3. L-Lipschitz continuous on C if there exists a constant $L \in (0, \infty)$ such that for all $x, y \in C$, we have

$$\|A(x) - A(y)\| \leq L\|x - y\|.$$

If $L = 1$, then A is called a nonexpansive mapping.

Lemma 2.1. [13] *Let* $\{a_n\}, \{b_n\} \subset (0, \infty)$ *be two sequences satisfying*

$$a_{n+1} \leq a_n + b_n, \ \sum_{n=0}^{\infty} b_n < \infty.$$

Then the sequence $\{a_n\}$ *is convergent.*

Denote by $\mathrm{Sol}(A, C)$ the solution set of $\mathrm{VI}(A, C)$. In this paper, we always assume that the solution set of $\mathrm{VI}(A, C)$ is not empty.

3. Main Results

Assumption 3.1. We consider the problem VI(A,C) under the following conditions:

(A1) The mapping A is pseudomonotone on C;

(A2) The mapping A Lipschitz continuous on C (with unknown Lipschitz constant).

To solve this problem, the following algorithm is proposed.

Algorithm 3.1.
Step 0. Choose $y^{-1}, y^0, x^0 \in C$; $\lambda_0 > 0, \delta \in (0,1)$, $\rho \in (0, \frac{1}{3})$. Set $n = 0$.
Step 1. Given λ_n, y^n and x^n. Compute

$$x^{n+1} = P_C(x^n - \lambda_n A(y^n)),$$
$$y^{n+1} = P_C(x^{n+1} - \lambda_n A(y^n)).$$

If $\lambda_n \|A(y^{n-1}) - A(y^n)\| \le \rho \|y^{n-1} - y^n\|$ **then** set $\lambda_{n+1} := \lambda_n$ **else** set $\lambda_{n+1} := \lambda_n \delta$.
Step 2. If $x^{n+1} = y^n$, **then** STOP, **else** update $n := n+1$ and GOTO **Step 1**.

Suppose now Algorithm 3.1 generates an infinite sequence $\{x^n\}$. We will prove that this sequence converges to a desired solution.

Theorem 3.1. *If the conditions* (A1)-(A2) *in Assumption 3.1 are satisfied. Then, the sequence* $\{x^n\}$ *generated by Algorithm 3.1 converges to a solution* x^* *of* VI(A,C).

Proof. Since $\delta \in (0,1)$, the sequence $\{\lambda_n\}$ is nonincreasing. We will show that this sequence is bounded away from zero. Indeed, we assume that $\lim_{n\to\infty} \lambda_n = 0$. Then, there exists a subsequence $\{\lambda_{n_i}\} \subset \{\lambda_n\}$ satisfying

$$\lambda_{n_i-1} \left\|A(y^{n_i-2}) - A(y^{n_i-1})\right\| > \rho \left\|y^{n_i-2} - y^{n_i-1}\right\|.$$

Let L be the Lipschitz constant of A, we have

$$\lambda_{n_i-1} > \rho \frac{\left\|y^{n_i-2} - y^{n_i-1}\right\|}{\|A(y^{n_i-2}) - A(y^{n_i-1})\|} \ge \frac{\rho}{L},$$

which contradicts the fact $\lim_{n\to\infty} \lambda_n = 0$. Thus, there exists a number $n_0 > 0$ such that

$$\lambda_{n+1} = \lambda_n \text{ and } \lambda_n \|A(y^{n-1}) - A(y^n)\| \le \rho \|y^{n-1} - y^n\| \quad \forall n \ge n_0. \tag{4}$$

Since

$$x^{n+1} = P_C(x^n - \lambda_n A(y^n)),$$

using the basic property of the projection, we have

$$\left\langle z - x^{n+1}, x^n - \lambda_n A(y^n) - x^{n+1} \right\rangle \le 0,$$

or equivalently,

$$\left\langle x^{n+1} - z, x^{n+1} - x^n \right\rangle \le \lambda_n \left\langle z - x^{n+1}, A(y^n) \right\rangle \quad \forall z \in C. \tag{5}$$

Analogously, from the definition of y^{n+1}, we obtain

$$\left\langle y^{n+1} - z, y^{n+1} - x^{n+1} \right\rangle \le \lambda_n \left\langle z - y^{n+1}, A(y^n) \right\rangle \quad \forall z \in C. \tag{6}$$

We have

$$\begin{aligned}\|x^{n+1}-z\|^2 &= \|y^n-z\|^2-\|x^{n+1}-y^n\|^2+2\langle x^{n+1}-y^n,x^{n+1}-z\rangle\\ &= \|x^n-z\|^2-\|x^n-y^n\|^2+2\langle y^n-x^n,y^n-z\rangle-\|x^{n+1}-y^n\|^2\\ &\quad+2\langle x^{n+1}-y^n,x^{n+1}-z\rangle\\ &= \|x^n-z\|^2-\|x^n-y^n\|^2-\|x^{n+1}-y^n\|^2\\ &\quad+2\langle y^n-x^n,y^n-x^{n+1}\rangle+2\langle x^{n+1}-x^n,x^{n+1}-z\rangle.\end{aligned}$$

Using (5) and (6), we have

$$\begin{aligned}\|x^{n+1}-z\|^2 &\le \|x^n-z\|^2-\|x^n-y^n\|^2-\|x^{n+1}-y^n\|^2\\ &\quad+2\lambda_n\langle A(y^{n-1}),x^{n+1}-y^n\rangle+2\lambda_n\langle A(y^n),z-x^{n+1}\rangle\\ &= \|x^n-z\|^2-\|x^n-y^n\|^2-\|x^{n+1}-y^n\|^2\\ &\quad+2\lambda_n\langle A(y^{n-1}),x^{n+1}-y^n\rangle+2\lambda_n\langle A(y^n),z-y^n\rangle+2\lambda_n\langle A(y^n),y^n-x^{n+1}\rangle.\end{aligned}$$

Let $z=x^*\in \mathrm{Sol}(A,C)$, due to the pseudomonotonicity of A, we obtain

$$\begin{aligned}\|x^{n+1}-x^*\|^2 &\le \|x^n-x^*\|^2-\|x^n-y^n\|^2-\|x^{n+1}-y^n\|^2\\ &\quad+2\lambda_n\langle A(y^{n-1})-A(y^n),x^{n+1}-y^n\rangle.\end{aligned}$$

From (4), for all $n\ge n_0$, we have

$$\begin{aligned}\|x^{n+1}-x^*\|^2 &\le \|x^n-x^*\|^2-\|x^n-y^n\|^2-\|x^{n+1}-y^n\|^2\\ &\quad+\rho\left(\|y^{n-1}-y^n\|^2+\|x^{n+1}-y^n\|^2\right)\\ &\le \|x^n-x^*\|^2-\|x^n-y^n\|^2-\|x^{n+1}-y^n\|^2\\ &\quad+\rho\left(2\|y^{n-1}-x^n\|^2+2\|x^n-y^n\|^2+\|x^{n+1}-y^n\|^2\right).\end{aligned}\tag{7}$$

From (7), we obtain

$$\begin{aligned}\|x^{n+1}-x^*\|^2+2\rho\|x^{n+1}-y^n\|^2 &\le \|x^n-x^*\|^2+2\rho\|y^{n-1}-x^n\|^2\\ &\quad-(1-2\rho)\|x^n-y^n\|^2-(1-3\rho)\|x^{n+1}-y^n\|^2.\end{aligned}\tag{8}$$

Since $\rho\in(0,\frac{1}{3})$, the sequence $\{\|x^n-x^*\|^2+2\rho\|y^{n-1}-x^n\|^2\}$ is decreasing and nonnegative. Therefore, it converges and the sequence $\{x^n\}$ is bounded. There exists the finite limit

$$\lim_{n\to\infty}\|x^n-x^*\|^2+2\rho\|y^{n-1}-x^n\|^2.$$

Morever, we infer that

$$\|x^{n+1}-y^n\|\to 0 \text{ and } \|y^n-x^n\|\to 0.\tag{9}$$

Therefore, there exists the finite limit

$$\lim_{n\to\infty}\|x^n-x^*\|=c.$$

The sequence $\{x^n\}$ is bounded. Therefore, there exists a subsequence $\{x^{n_j}\}\subset\{x^n\}$ such that

$x^{n_j} \to \bar{x} \in C$. We will prove that $\bar{x} \in \mathrm{Sol}(A,C)$. Indeed, from (5), for all $z \in C$, $n_j \geq k_0$, we get

$$\langle x^{n_j+1} - z, x^{n_j+1} - x^{n_j} \rangle \leq \lambda_{n_0} \langle z - x^{n_j+1}, A(y^{n_j}) \rangle. \tag{10}$$

Using (9), we have

$$\|x^{n+1} - x^n\| \leq \|x^{n+1} - y^n\| + \|y^n - x^n\| \to 0. \tag{11}$$

In (10), letting $j \to \infty$, using the continuity of A, (9) and (11), we have

$$\langle A(\bar{x}), z - \bar{x} \rangle \geq 0 \ \forall z \in C,$$

which means $\bar{x} \in \mathrm{Sol}(A,C)$. Since the sequence $\{\|x^n - \bar{x}\|\}$ is convergent, we have

$$\lim_{n\to\infty} \|x^n - \bar{x}\| = \lim_{j\to\infty} \|x^{n_j} - \bar{x}\| = 0.$$

We present a numerical example to give some insight into the behavior of the proposed algorithm. We also compare our algorithm with the Popov's algorithm. We perform the iterative schemes in MATLAB codes under version MATLAB R2021A and run on a PC with CPU i5 5200U and 6GB RAM.

Example 3.1. Let $C = \left\{x \in \mathbb{R}^M : -5 \leq x_i \leq 5 \ \ \forall i \in 1, \ldots, M\right\}$. We take $A(x) = F(x)$, with F is a randomly generated positive definite $M \times M$ matrix. The starting point y^{-1}, y^0 and x^0 are generated randomly. We use the stopping rule $\|x^n - x^*\| \leq 10^{-3}$, with $x^* = (0, \ldots, 0)^\top$ is the exact solution of the considered problem.

The parameters are chosen by the grid search method with the grid size of 0.1 as following:

- In Algorithm 3.1, we choose $\rho = 0.3, \delta = 0.9, \lambda_{-1} = 3.5$;
- In Popov's algorithm, we choose $\lambda = \frac{0.3}{L}$.

The results are shown in Figure 1 and Table 1. Figure 1 shows the change of $error = \|x^n - x^*\|$ of two algorithms according to time. We can see that our algorithm shows a better behaviour in term of computational time.

Table 1. *Compare Algorithm 3.1 with Popov's Alg. in Example 3.1*

	Popov's Alg.		**Alg. 3.1**	
	Times(s)	**Iter.**	**Times(s)**	**Iter.**
$M = 50$	0.0024	786	0.0019	216
$M = 100$	0.0270	2235	0.0097	318
$M = 200$	0.0724	3777	0.0254	845
$M = 500$	0.3784	5788	0.0726	1290

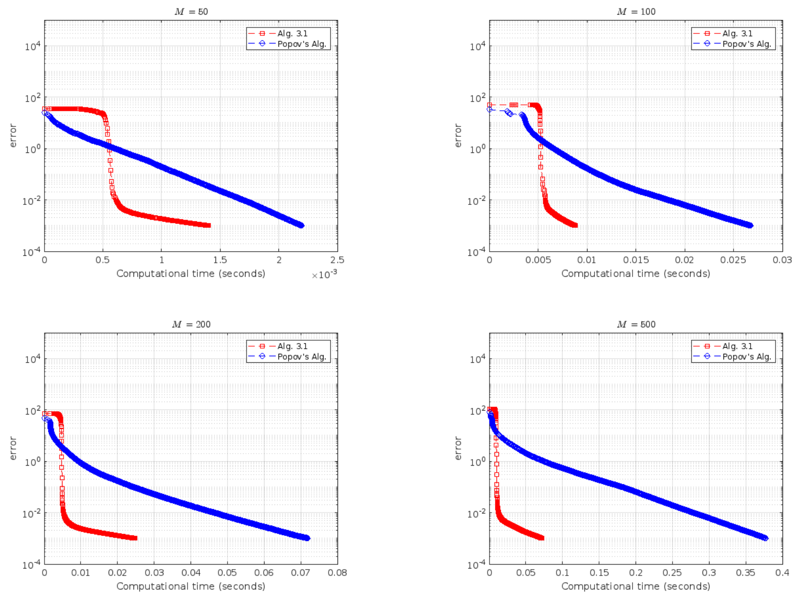

Figure 1. *Performance of two algorithms in Example 3.1*

4. Conclusion

In this paper, we have presented the iterative method to solve pseudomonotone variational inequalities. Our algorithm can be applied for variational inequalities with unknown Lipschitz continuous modulus. The algorithm does not require the step-sizes tending to zero. Moreover, the new algorithm only requires to compute the value of the mapping A at a point. These features help to greatly reduce the computational cost of the proposed algorithm.

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