A SELF-ADAPTIVE ITERATIVE ALGORITHM FOR SOLVING A CLASS OF BILEVEL SPLIT VARIATIONAL INEQUALITY PROBLEM IN HILBERT SPACES

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ABSTRACT

The aim of this paper is to examine a complex bilevel problem of finding solutions to multiple sets of variational inequalities in Hilbert spaces. This problem is challenging and requires an effective solution method. We propose a self-adaptive algorithm that dynamically adjusts the step sizes based on the information from the previous step. We rigorously prove the strong convergence of our algorithm and demonstrate that it requires less restrictive conditions than the ones used by Censor et al. (2012). We apply our main results to solve an important problem: the split variational inequality problem. Our analysis shows that our algorithm converges strongly with weaker assumptions than those used recently by Censor et al. (2012) and Buong (2017). To further illustrate the convergence behavior of our method, we provide some numerical examples that demonstrate its effectiveness and efficiency.

THUẬT TOÁN LẶP TỰ THÍCH NGHI GIẢI BÀI TOÁN BẤT ĐẲNG THỨC BIẾN PHÂN HAI CẤP TRONG KHÔNG GIAN HILBERT

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TỪ KHÓA

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Bất đẳng thức biến phân
Không gian Hilbert
Ánh xạ không giãn
Toán tử chiếu

TÓM TẮT

Mục tiêu của bài báo này là nghiên cứu về nghiệm của bài toán bất đẳng thức biến phân hai cấp đa tập trong không gian Hilbert. Đây là một bài toán khó và đòi hỏi một phương pháp giải quyết hiệu quả. Chúng tôi đề xuất một thuật toán lặp tự thích nghi, cho phép cỡ bước tự điều chỉnh dựa trên thông tin từ bước lặp trước đó. Chúng tôi chứng minh sự hội tụ mạnh của thuật toán và sử dụng điều kiện nhẹ hơn so với điều kiện của Censor và cộng sự (2012). Phân tích của chúng tôi cho thấy thuật toán hội tụ mạnh với các giả thiết yếu hơn so với các giả thiết được sử dụng gần đây bởi Censor và cộng sự (2012) và Buong (2017). Để minh họa thêm về sự hội tụ của phương pháp, chúng tôi đưa ra ví dụ số minh họa để chứng minh hiệu quả của nó.

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1. Introduction

Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two real Hilbert spaces with inner product $\langle .,. \rangle$ and norm $\|.\|$. Let C be a closed convex subset of $\mathscr{H}_1$, Q be a closed convex subset of $\mathscr{H}_2$, and let $F : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear operator, and given single-valued operators $A : C \to \mathscr{H}_1$ and $B : Q \to \mathscr{H}_2$. The split variational inequality problem introduced first by Censor et al. [1] is to 228(10): 491 - 499

$$\text{find } u^* \in \Omega_{\text{VIP}} := S_{(A,C)} \bigcap F^{-1}(S_{(B,Q)}), \tag{SVIP}$$

where $S_{(C,A)}$ denotes as the set of all solutions of the following variational inequality problem, for short VIP(A,C) [2], [3],

$$\langle Au^*, u - u^* \rangle \geq 0 \ \ \forall u \in C.$$

To solve the (SVIP), Censor et al. [1] proposed the following algorithm

$$\begin{cases} x^1 \in \mathscr{H}_1, \ \text{any element}, \\ x^{n+1} = x^n + \gamma((P_C(I^{\mathscr{H}_1} - \lambda A) - I^{\mathscr{H}_1})x^n + F^*(P_Q(I^{\mathscr{H}_2} - \lambda B) - I^{\mathscr{H}_2})Fx^n), \quad n \geq 1. \end{cases}$$

They presented a weak convergence result when A and B are α-inverse strongly monotone operators, the parameters γ and λ satisfy the following conditions

$$\gamma \in \left(0, \frac{1}{1 + \|F\|^2}\right); \tag{F1}$$

$$\lambda \in (0, 2\alpha]. \tag{L1}$$

Another special case of the (SVIP) is the split feasibility problem

$$\text{find } u^* \in \Omega_{\text{SFP}} := C \bigcap F^{-1}Q \tag{SFP}$$

which had already been studied and used in practice as a model in intensitymodulated radiation therapy (IMRT) treatment planning; see [4]. To solve the (SVIP), Censor et al. [1] proposed a weakly convergent projection method

$$\begin{cases} x^1 \in \mathscr{H}_1, \ \text{any element}, \\ x^{n+1} = P_C(I^{\mathscr{H}_1} - \lambda A)\left(x^n + \gamma F^*(P_Q(I^{\mathscr{H}_2} - \lambda B) - I^{\mathscr{H}_2})(Fx^n)\right), \quad n \geq 1, \end{cases}$$

involving α_1, α_2-inverse strongly monotone mappings A and B, the parameter λ satisfies the condition (L1) with $\alpha := \min\{\alpha_1, \alpha_2\}$, and the parameter γ satisfies

$$\gamma \in \left(0, \frac{1}{L}\right), \tag{F2}$$

where L is the spectral radius of the operator F^*F.

In the present paper, inspired by the above mentioned works, we suggest and analyze an iterative method for solving the following bilevel split variational inequality problem

$$\text{find } x^* \in \Omega_{\text{SVIP}} \text{ such that } \langle (I^{\mathscr{H}_1} - G)x^*, x - x^* \rangle \geq 0 \ \ \forall x \in \Omega_{\text{SVIP}}, \tag{BVIP}$$

where $G : \mathscr{H}_1 \to \mathscr{H}_1$ is a contraction mapping. Our algorithms are designed by using cyclic iterative methods [5, 6] and overcome the disadvantages of the CQ–algorithm [7, 8]. Namely, our algorithms give strong convergence. In each iteration of the new algorithms, we do not need to compute norm of operator F. Also, the new algorithms do not require to know the Lipschitz constant l_A of the involving mapping. Morever, our algorithm uses dynamic step-sizes, chosen based on information of the previous step. All these features help to reduce the computational cost and speed up our algorithms.

The remaining part of this paper is organized as follows: the next section displays some lemmas that will be used for the validity and convergence of the algorithm. The third section is devoted to the description of our proposed algorithm and its strong convergence result. Finally, we illustrate the proposed method by considering some numerical experiments.

2. Preliminaries

In this section, we introduce some mathematical symbols, definitions, and lemmas which can be used in the proof of our main result.

Let $\mathscr{H}, \mathscr{H}_1$, and $\mathscr{H}_2$ be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$. In what follows, we write $x^k \rightharpoonup x$ to indicate that the sequence $\{x^k\}$ converges weakly to x while $x^k \to x$ indicate that the sequence $\{x^k\}$ converges strongly to x.

It is well known that for all $x, y \in \mathscr{H}$,

$$2\langle x,y\rangle = \|x+y\|^2 - \|x\|^2 - \|y\|^2 = \|x\|^2 + \|y\|^2 - \|x-y\|^2. \tag{1}$$

Let $G : C \to C$ be mapping. G is said to be l-Lipschitz continuous if

$$\|Gx - Gy\| \le l\|x-y\| \quad \forall x, y \in C,$$

where l is a positive constant. G is said to be contraction operator, if $l \in [0,1)$, and nonexpansive, if $l = 1$. We denote the set of fixed points of G by Fix(G), that is, $\mathrm{Fix}(G) = \{x \in C \mid Gx = x\}$.

Let C be a nonempty, closed and convex subset of $\mathscr{H}$. We know that, for each $x \in \mathscr{H}$, there is a unique $P_C x \in C$ such that

$$\|x - P_C x\| = \inf_{u \in C} \|x - u\|, \tag{2}$$

and the mapping $P_C : \mathscr{H} \to C$ defined by (2) is called the metric projection from $\mathscr{H}$ onto C.

Lemma 2.1 (see [9]). *Let P_C is the metric projection from $\mathscr{H}$ onto C. Then, we have*

(i) *P_C is a nonexpansive operator of $\mathscr{H}$ onto C.*
(ii) *$P_C x \in C$ for all $x \in \mathscr{H}$ and if $x \in C$, then $P_C x = x$.*
(iii) *For given $x \in \mathscr{H}$, and $y \in C$, $y = P_C x$ if and only if $\langle x - y, z - y\rangle \le 0 \quad \forall z \in C$.*

Mapping $A : \mathscr{H}_1 \to \mathscr{H}_1$ is said to be η_A-strongly monotone on $\mathscr{H}_1$ if there exists $\eta_A > 0$ such that

$$\langle Ax - Ay, x - y\rangle \ge \eta_A \|x-y\|^2 \quad \forall x, y \in \mathscr{H}_1.$$

It is easy to see that if G is a contraction mapping with the contraction coefficient $\tau \in [0,1)$, $I^{\mathscr{H}_1} - G$ is l-Lipschitz continuous and η-strongly monotone operator on $\mathscr{H}_1$ with $l = (1+\tau)$ and $\eta = (1-\tau)$. So, if Ω is a nonempty closed convex subset of $\mathscr{H}_1$, then the VIP($I^{\mathscr{H}_1} - G, \Omega$) has a unique solution. Morever, from Lemma 2.1(iii), we have that the point $x^* \in \mathscr{H}_1$ is a solution of the VIP($I^{\mathscr{H}_1} - G, \Omega$) if and only if $x^* = P_\Omega G x^*$.

Lemma 2.2 (see [1]). *Let $A : C \to \mathscr{H}$ be η-inverse strongly monotone on C and $\lambda > 0$ be a constant satisfying $0 < \lambda \le 2\eta$. Define the mapping $V : C \to C$ by taking*

$$Vx = P_C(I^{\mathscr{H}} - \lambda A)x \quad \forall x \in C. \tag{3}$$

Then V is nonexpansive on C, furthermore, $\mathrm{Fix}(V) = S_{(A,C)}$.

Let $F : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear operator. An operator $F^* : \mathscr{H}_2 \to \mathscr{H}_1$ with the property $\langle Fx, y\rangle = \langle x, F^* y\rangle$ for all $x \in \mathscr{H}_1$ and $y \in \mathscr{H}_2$, is called an adjoint operator. The adjoint operator of a bounded linear operator F on a Hilbert space always exists and is uniquely determined. Furthermore, F^* is a bounded linear operator.

The following lemmas are used in the sequel in the proofs of the main results of our paper.

Lemma 2.3 (Opial' Lemma, [10]). *Let C be a nonempty closed convex subset of a real Hilbert space $\mathscr{H}$ and $T : C \to C$ be a nonexpansive operator with Fix(T) $\neq \varnothing$. If $\{x^n\}$ is a sequence in C converging weakly to x^* and if the sequence $\{(I^{\mathscr{H}} - T)x^n\}$ converges strongly to y, then $(I^{\mathscr{H}} - T)x^* = y$; in particular, if $y = 0$, then* $x \in \mathrm{Fix}(T)$.

Lemma 2.4 (see [11]). *Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the condition $s_{n+1} \le (1-b_n)s_n + b_n c_n$, $n \ge 0$, where $\{b_n\}$ and $\{c_n\}$ are sequences of real numbers such that*

(*i*) $\{b_n\} \subset (0,1)$ *for all* $n \geq 0$ *and* $\sum_{n=1}^{\infty} b_n = \infty$,
(*ii*) $\limsup_{n\to\infty} c_n \leq 0$.

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.5 (Maingé, [12]). *Let* $\{s_n\}$ *be a real sequence of real numbers. Assume* $\{s_n\}$ *does not decrease at infinity, that is, there exists at least a subsequence* $\{s_{n_k}\}$ *of* $\{s_n\}$ *such that* $s_{n_k} \leq s_{n_k+1}$ *forallk* ≥ 0. *For any* $n \geq n_0$, *define an integer sequence* $\{\nu(n)\}$ *as* $\nu(n) := \max\{n_0 \leq k \leq n \mid s_k < s_{k+1}\}$, $k \geq k_0$. *Then* $\nu(n) \to \infty$ *as* $k \to \infty$ *and for all* $n \geq n_0$, *we have* $\max\{s_{\nu(n)}, s_n\} \leq s_{\nu(n)+1}$.

3. Results and Discussion

We consider the (SVIP) under the following conditions.

Assumption 3.1.

(A1) $A : \mathscr{H}_1 \to \mathscr{H}_1$ is η_A-inverse strongly monotone on $\mathscr{H}_1$.
(A2) $B : \mathscr{H}_2 \to \mathscr{H}_2$ is η_B-inverse strongly monotone on $\mathscr{H}_2$.
(A3) $F : \mathscr{H}_1 \to \mathscr{H}_2$ be a bounded linear operator.
(A4) $G : \mathscr{H}_1 \to \mathscr{H}_1$ is a contraction operator with the contraction coefficient $\tau \in [0,1)$.
(A5) The solution set Ω_{SVIP} of (SVIP) is not empty.

Our algorithm can be expressed as follows.

Algorithm 1

Step 0. Select the initial point $x^1 \in \mathscr{H}_1$ and the sequences $\{\rho_n\} \subset [a,b] \subset (0,1)$, $\{e_n\} \subset [c,d] \subset (0,+\infty)$, and the sequences $\{\alpha_n\}$ and λ such that the conditions

$$\{\alpha_n\} \subset (0,1),\ \alpha_n \to 0 \text{ as } n \to \infty, \text{ and } \sum_{n=1}^{\infty} \alpha_n = \infty; \tag{C1}$$

and (L1) with $\alpha := \min\{\eta_{A_i}, \eta_{B_j} \mid i = 1,\ldots,M,\ j = 1,\ldots,N\}$.
Set $k := 1$.
Step 1. Compute $y^n = P_C(I^{\mathscr{H}_1} - \lambda A)x^n$ and let $\kappa_n^1 = \|y^n - x^n\|$.
Step 2. Compute $z^n = P_Q(I^{\mathscr{H}_2} - \lambda B)Fx^n$ and let $\kappa_n^2 = \|z^n - Fx^n\|$.
Step 3. Let $D_n := \max\{\kappa_n^1, \kappa_n^2\}$. If $D_n = \kappa_n^1$ then put $v^n := y^n$ and $T := I^{\mathscr{H}_1}$. Else, put $v^n := z^n$ and $T := F$.
Step 4. Compute $u^n = x^n - \gamma_n T^*(Tx^n - v^n)$, where the step size γ_n is defined by

$$\gamma_n = \rho_n \frac{\|Tx^n - v^n\|^2}{\|T^*(Tx^n - v^n)\|^2 + e_n}. \tag{F}$$

Step 5. Compute $x^{n+1} = \alpha_n Gx^n + (1-\alpha_n)u^n$.
Step 6. Set $n := n+1$ and go to **Step 1**.

The following theorem shows the validity and convergence of the algorithm.

Theorem 3.1. *Suppose that all conditions in Assumption 3.1 are satisfied. Then the sequence* $\{x^n\}$ *generated by Algorithm 1 converges strongly to the unique solution of the* $\text{VIP}(I^{\mathscr{H}_1} - G, \Omega_{\text{SVIP}})$.

Proof. Since G is a contraction mapping, $P_{\Omega_{\text{SVIP}}}G$ is a contraction too. By Banach contraction operator principle, there exists a unique point $u^* \in \Omega_{\text{SVIP}}$ such that $P_{\Omega_{\text{SVIP}}}Gu^* = u^*$. By Lemma 1(*iii*), we obtain u^* is the unique solution to the $\text{VIP}(I^{\mathscr{H}_1} - G, \Omega_{\text{SVIP}})$.

1. First, we prove that the sequence $\{x^n\}$ in Algorithm 1 is bounded.

Let $u \in \Omega_{\text{SVIP}}$. Hence, $u \in S_{(A,C)}$ and $Fu \in S_{(B,Q)}$. We consider the following two cases.

Case 1a. $D := \kappa^2$.

Since $Fu \in S_{(B,Q)}$ it follows from Lemma 2.2 that $Fu = P_Q\big(I^{\mathscr{H}_2} - \lambda B\big)Fu$. Another, since B is an η_B-inverse strongly monotone operator, $I^{\mathscr{H}_2} - \lambda B$ is a nonexpansive mapping for $\lambda \in [0; 2\eta_B)$. So $P_Q\big(I^{\mathscr{H}_2} - \lambda B\big)$ is a nonexpansive mapping. Now, from Step 2, 3, and 4 in Algorithm 1, the property of adjoint operator F^*, and (1), we have that

$$\begin{aligned}
\|u^n-u\|^2 &= \|x^n-\gamma_n F^*[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n-u\|^2\\
&= \|x^n-u\|^2+\gamma_n^2\|F^*[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^2-2\gamma_n\langle Fx^nFu,[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\rangle\\
&= \|x^n-u\|^2+\gamma_n^2\|F^*[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^2\\
&\qquad -\gamma_n\Big(\|Fx^n-Fu\|^2+\|[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^2-\|P_Q(I^{\mathscr{H}_2}-\lambda B)Fx^n-Fu\|^2\Big)\\
&= \|x^n-u\|^2+\gamma_n^2\|F^*[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^2+\gamma_n\Big(\|P_Q(I^{\mathscr{H}_2}-\lambda B)Fx^n-P_Q(I^{\mathscr{H}_2}-\lambda B)Fu\|^2\\
&\qquad -\|[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^2-\|Fx^n-Fu\|^2\Big)\\
&\le \|x^n-u\|^2+\gamma_n^2\|F^*[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^2+\gamma_n\Big(\|Fx^n-Fu\|^2-\|Fx^n-Fu\|^2\\
&\qquad -\|[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^2\Big)\\
&= \|x^n-u\|^2+\gamma_n^2\|F^*[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^2-\gamma_n\|[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^2.
\end{aligned}$$

From the last inequality and (F) we obtain

$$\begin{aligned}
\|u^n-u\|^2 &\le \|x^n-u\|^2+\rho_n^2\frac{\|[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^4}{\Big(\|F^*[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^2+e_n\Big)^2}\times\|F^*[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^2\\
&\qquad -\rho_n\frac{\|[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^4}{\|F^*[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^2+e_n}\\
&\le \|x^n-u\|^2+\rho_n^2\frac{\|[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^4}{\|F^*[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^2+e_n}-\rho_n\frac{\|[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^4}{\|F^*[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^2+e_n}\\
&= \|x^n-u\|^2-\rho_n(1-\rho_n)\frac{\|[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^4}{\|F^*[I^{\mathscr{H}_2}-P_Q(I^{\mathscr{H}_2}-\lambda B)]Fx^n\|^2+e_n}. \qquad (4)
\end{aligned}$$

Case 1b. $D := \kappa^1$.

It follows from Steps 1, 3, and 4 in Algorithm 1 that

$$\|u^n-u\|^2 = \|x^n-\gamma_n[I^{\mathscr{H}_1}-P_C(I^{\mathscr{H}_1}-\lambda A)]x^n-u\|^2,$$

and we also get

$$\|u^n-u\|^2 \le \|x^n-u\|^2-\rho_n(1-\rho_n)\frac{\|[I^{\mathscr{H}_1}-P_C(I^{\mathscr{H}_1}-\lambda A)]x^n\|^4}{\|[I^{\mathscr{H}_1}-P_C(I^{\mathscr{H}_1}-\lambda A)]x^n\|^2+e_n}. \qquad (5)$$

It follows from the convexity of the norm function on $\mathscr{H}_1$, the contraction property of G with the contraction coefficient $\tau\in[0,1)$, (4), (5), and Step 5 in Algorithm 1 that

$$\begin{aligned}
\|x^{n_1}-u\| &= \|\alpha_n(Gx^n-u^*)+(1-\alpha_n)(u^n-u)\|\\
&\le \alpha_n\Big(\|Gx^n-Gu\|+\|Gu-u\|\Big)+(1-\alpha_n)\|u^n-u\|\\
&\le \alpha_n\Big(\|Gx^n-Gu\|+\|Gu-u\|\Big)+(1-\alpha_n)\|x^n-u\|\\
&\le \tau\alpha_n\|x^n-u\|+\alpha_n\|Gu-u\|+(1-\alpha_n)\|u^n-u\|\\
&= [1-(1-\tau)\alpha_n]\|x^n-u\|+(1-\tau)\alpha_n\frac{\|Gu-u\|}{1-\tau}\\
&\le \max\left\{\|x^n-u\|,\frac{\|Gu-u\|}{1-\tau}\right\}\\
&\;\;\vdots
\end{aligned}$$

$$\leq \max\left\{\|x^1 - u\|, \frac{\|Gu - u\|}{1-\tau}\right\}.$$

This implies that the sequence $\{x^n\}$ is bounded. Since $P_C\big(I^{\mathscr{H}_1} - \lambda A\big)$ and $P_Q\big(I^{\mathscr{H}_2} - \lambda B\big)$ are nonexpansive mappings and F is the bounded linear operator, we also have the sequences $\{y^n\}$ and $\{z^n\}$ are bounded. This combines with the boundedness of the sequence $\{e_n\}$, we get that

$$\widehat{M} := \max\big\{\sup_n \|(I^{\mathscr{H}_1} - P_C(I^{\mathscr{H}_1} - \lambda A))x^n\|^2 + e_n, \sup_n \|F^*(I^{\mathscr{H}_2} - P_Q(I^{\mathscr{H}_2} - \lambda B))Fx^n\|^2 + e_n\big\} < \infty.$$

2. Next, we prove that

$$\|x^{n+1} - u\|^2 \leq [1 - (1-\tau)\alpha_n]\|x^n - u\|^2 + 2\alpha_n\langle Gu - u, x^{n+1} - u\rangle. \tag{6}$$

Indeed, it follows from Steps 1–3 in Algorithm 1, (4), (5), and $\{e_n\} \subset [c,d] \subset (0,\infty)$, that

$$D^4 \leq \frac{\widehat{M}}{\rho_n(1-\rho_n)}\Big(\|x^n - u\|^2 - \|u^n - u\|^2\Big). \tag{7}$$

at nth step iteration. It follows from the convexity of the function $\|.\|^2$, Step 5 in Algorithm 1, the condition (C1), and (7) that

$$\begin{aligned}
\|x^{n+1} - u\|^2 &= \|\alpha_n(Gx^n - u) + (1-\alpha_n)(u^n - u)\|^n \\
&\leq \alpha_n\|Gx^n - u\|^n + (1-\alpha_n)\|u^n - u\|^2 \\
&\leq \alpha_n\|Gx^n - u\|^n + \|u^n - u\|^2 \\
&\leq \alpha_n\|Gx^n - u\|^n + \|x^n - u\|^2 - \frac{\rho_n(1-\rho_n)}{\widehat{M}}D^4,
\end{aligned}$$

which implies that

$$D^4 \leq \frac{\widehat{M}}{\rho_n(1-\rho_n)}\Big(\|x^n - u\|^2 - \|x^{n+1} - u\|^2 + \alpha_n\|Gx^n - u\|^2\Big). \tag{8}$$

From Step 5 in Algorithm 1 and the contraction property of G with the contraction coefficient $\tau \in [0,1)$, we have that

$$\begin{aligned}
\|x^{n+1} - u\|^2 &= \langle \alpha_n(Gx^n - u) + (1-\alpha_n)(u^n - u), x^{n+1} - u\rangle \\
&= (1-\alpha_n)\langle u^n - u, x^{n+1} - u\rangle + \alpha_n\langle Gx^n - u, x^{n+1} - u\rangle \\
&\leq \frac{1-\alpha_n}{2}\Big(\|u^n - n\|^2 + \|x^{n+1} - u\|^2\Big) \\
&\quad + \alpha_n\langle Gx^n - Gu, x^{n+1} - u\rangle + \alpha_n\langle Gu - u, x^{n+1} - u\rangle \\
&\leq \frac{1-\alpha_n}{2}\Big(\|u^n - n\|^2 + \|x^{n+1} - u\|^2\Big) \\
&+ \frac{\alpha_n}{2}\Big(\tau\|x^n - u\|^2 + \|x^{n+1} - u\|^2\Big) + \alpha_n\langle Gu - u, x^{n+1} - u\rangle.
\end{aligned}$$

This implies that

$$\|x^{n+1} - u\|^2 \leq (1-\alpha_n)\|u^n - u\| + \alpha_n\tau\|x^n - u\|^2 + 2\alpha_n\langle Gu - u, x^{n+1} - u\rangle.$$

From (4), (5), and the last inequality, we obtain (6).

3. Now we claim that $\lim_{n\to\infty}\|x^n - u^*\| = 0$, where u^* is the unique solution of the VIP$(I^{\mathscr{H}_1} - G, \Omega)$, that is, $u^* = P_\Omega G u^*$.

Indeed, it follows from (6) with u replaced by u^* that

$$\|x^{n+1} - u^*\|^2 \leq [1 - (1-\tau)\alpha_n]\|x^n - u^*\|^2 + (1-\tau)\alpha_n\left[\frac{2}{1-\tau}\langle Gu^* - u^*, x^{n+1} - u^*\rangle\right], \ \forall n \geq 1. \tag{9}$$

We consider two cases.

Case 3a. There exists an integer $n_0 \geq 0$ such that $\|x^{n+1} - u^*\| \leq \|x^n - u^*\|$ for all $n \geq n_0$.

Then, $\lim_{n\to\infty}\|x^n - u^*\|$ exists. From the boundedness of the sequence $\{Gx^n\}$, the conditions (C1) and $\{\rho_n\} \subset [a,b] \subset (0,1)$, it follows from (8) with u replaced by u^* that $D_n \to 0$. Since the definition of D, the sequences $\{\kappa^1\}$ and $\{\kappa^2\}$ in Algorithm 1 also converge to 0. This implies that

$$\lim_{n\to\infty}\|(I^{\mathscr{H}_1} - P_C(I^{\mathscr{H}_1} - \lambda A)x^n\| = 0 \tag{10}$$

and

$$\lim_{n\to\infty}\|(I^{\mathscr{H}_2} - P_Q(I^{\mathscr{H}_2} - \lambda B)Fx^n\| = 0. \tag{11}$$

From Step 4 in Algorithm 1, (10), and (11), we obtain

$$\|x^n - u^n\| = \gamma_n\|T^*(Tx^n - v^n)\| \to 0 \text{ as } n \to \infty. \tag{12}$$

From the boundedness of the sequences $\{x^n\}, \{y^n\}, \{z^n\}$, Steps 4 and 5 in Algorithm 1, and the condition (C1), we also have

$$\|x^{n+1} - u^n\| = \alpha_n\|Gx^n - u^n\| \to 0 \text{ as } n \to \infty,$$

combining (12), we have

$$\|x^{n+1} - x^n\| \to 0 \text{ as } n \to \infty. \tag{13}$$

Now we show that $\limsup_{n\to\infty}\langle Gu^* - u^*, x^{n+1} - u^*\rangle \leq 0$. Indeed, suppose that $\{x^{n_k}\}$ is a subsequence of $\{x^n\}$ such that

$$\limsup_{n\to\infty}\langle Gu^* - u^*, x^n - u^*\rangle = \lim_{n\to\infty}\langle Gu^* - u^*, x^{n_k} - u^*\rangle. \tag{14}$$

Since $\{x^{n_k}\}$ is bounded, there exists a subsequence $\{x^{n_{k_l}}\}$ of $\{x^{n_k}\}$ which converges weakly to some point $u^\dagger$. Without loss of generality, we may assume that $x^{n_k} \rightharpoonup u^\dagger$. We shall prove that $u^\dagger \in \Omega$. Indeed, from Lemma 2.3 and (10) we obtain $u^\dagger \in \mathrm{Fix}(P_C(I^{\mathscr{H}_1} - \lambda A))$, that is $u^\dagger \in S_{(A,C)}$. Moreover, since each F is a bounded linear operator, $Fx^{n_k} \rightharpoonup Fu^\dagger$. Using Lemma 2.3 and (11), we also obtain $Fu^\dagger \in S_{(B,Q)}$. Hence,$u^\dagger \in F^{-1}(S_{(B,Q)})$. Consequently, $u^\dagger \in \Omega$. So, from $u^* = P_\Omega Gu^*$, (14), and Lemma 1(iii) we deduce that

$$\limsup_{n\to\infty}\langle Gu^* - u^*, x^n - u^*\rangle = \langle Gu^* - u^*, u^\dagger - u^*\rangle \leq 0,$$

which combined with (13) gives

$$\limsup_{n\to\infty}\langle Gu^* - u^*, x^{n+1} - u^*\rangle \leq 0. \tag{15}$$

Now, the inequality (9) can be rewritten in the form

$$\|x^{n+1} - u^*\|^2 \leq (1 - \widehat{b}_n)\|x^n - u^*\|^2 + \widehat{b}_n\widehat{c}_n,\ n \geq 1,$$

where

$$\widehat{b}_n = (1-\tau)\alpha_n \text{ and } \widehat{c}_n = \frac{2}{1-\tau}\langle Gu^* - u^*, x^{n+1} - u^*\rangle.$$

Since $\tau \in (0,1)$, $\{\alpha_n\} \subset (0,1)$, and $\sum_{n=1}^\infty \alpha_n = \infty$, $\{\widehat{b}_n\} \subset (0,1)$ and $\sum_{n=1}^\infty \widehat{b}_n = \infty$. Consequently, from $\tau \in (0,1)$ and (15), we have that $\limsup_{n\to\infty}\widehat{c}_n \leq 0$. Finally, by Lemma 2.4, $\lim_{n\to\infty}\|x^n - u^*\| = 0$.

Case 3b. There exists a subsequence $\{n_k\}$ of $\{n\}$ such that $\|x^{n_k} - u^*\| \leq \|x^{n_k+1} - u^*\|$ for all $k \geq 0$.

Hence, by Lemma 2.5, there exists an integer, nondecreasing sequence $\{\nu(n)\}$ for $n \geq n_0$ (for some n_0 large enough) sch that $\nu(n) \to \infty$ as $n \to \infty$,

$$\|x^{\nu(n)} - u^*\| \leq \|x^{\nu(n)+1} - u^*\| \text{ and } \|x^n - u^*\| \leq \|x^{\nu(n)+1} - u^*\| \tag{16}$$

for each $n \geq 1$. From (9) with n replaced by $\nu(n)$, we have

$$0 < \|x^{\nu(n)+1} - u^*\|^2 - \|x^{\nu\nu(n)} - u^*\|^2 \leq 2\alpha_{\nu(n)}\langle Gu^* - u^*, x^{\nu(n)+1} - u^*\rangle.$$

Since $\alpha_{\nu(n)} \to 0$ and the boundedness of $\{x^{\nu(n)}\}$, we conclude that

$$\lim_{n\to\infty}\left(\|x^{\nu(n)+1}-u^*\|^2-\|x^{\nu(n)}-u^*\|^2\right)=0. \tag{17}$$

By a similar argument to Case 3a, we obtain

$$\lim_{n\to\infty}\|(I^{\mathscr{H}^1}-P_C(I^{\mathscr{H}^1}-\lambda A))x^{\nu(n)}\|=0$$

and

$$\lim_{n\to\infty}\|(I^{\mathscr{H}^2}-P_Q(I^{\mathscr{H}^2}-\lambda B))Fx^{\nu(n)}\|=0.$$

Also we get

$$\|x^{\nu(n)+1}-u^*\|^2\le[1-(1-\tau)\alpha_{\nu(n)}]\|x^{\nu(n)}-u^*\|^2+2\alpha_{\nu(n)}\langle Gu^*-u^*,x^{\nu(n)+1}-u^*\rangle,$$

where, $\limsup_{n\to\infty}\langle Gu^*-u^*,x^{\nu(n)+1}-u^*\rangle\le 0$. Since the first inequality in (16) and $\alpha_{\nu(n)}>0$, we have that

$$(1-\tau)\|x^{\nu(n)}-u^*\|^2\le 2\langle Gu^*-u^*,x^{\nu(n)+1}-u^*\rangle.$$

Thus, from $\limsup_{n\to\infty}\langle Gu^*-u^*,x^{\nu(n)+1}-u^*\rangle\le 0$ and $\tau\in[0,1)$, we get $\lim_{n\to\infty}\|x^{\nu(n)}-u^*\|^2=0$. This together with (17) implies that $\lim_{n\to\infty}\|x^{\nu(n)+1}-u^*\|^2=0$. Which together with the second inequality in (16) implies that $\lim_{n\to\infty}\|x^n-u^*\|^2=0$. This completes the proof. □

We give a numerical experiment to illustrate the performance of our algorithm. This result is performed in Python running on a laptop Dell Inspiron 3593 Intel core i7, 1.30 GHz 8GB RAM.

Example 3.1. Let $\mathscr{H}_1=\mathbb{R}^3$ and $\mathscr{H}_2=\mathbb{R}^4$. Operators $A:\mathbb{R}^3\to\mathbb{R}^3$ and $B:\mathbb{R}^4\to\mathbb{R}^4$ are defined by

$$Ax=\begin{bmatrix}1&1&1\\1&2&2\\1&2&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix},\ x=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\in\mathbb{R}^3 \quad\text{and}\quad Bx=\begin{bmatrix}2&0&0&0\\0&3&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix},\ x=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\in\mathbb{R}^4$$

that are inverse strongly monotone operator with constant $\eta_A=\frac{2}{\sqrt{5}+17}$ and $\eta_B=\frac{1}{3}$. Bounded linear operator $F:\mathbb{R}^3\to\mathbb{R}^4$, $Fx=\begin{bmatrix}2&1&1\\1&0&0\\0&0&1\\0&1&-2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$. And $Gx:\mathbb{R}^3\to\mathbb{R}^3$, $Gx=\tau x$ is contractive operator with constant $\tau\in[0,1)$. Let C and Q are defined by

$$C=\{x\in\mathbb{R}^3,\langle a_1,x\rangle\le b_1\},\ \text{with } a_1=\begin{bmatrix}2&-1&1\end{bmatrix}^\top,\ b_1=1;$$
$$Q=\{x\in\mathbb{R}^4,\|x\|\le b_2\},\ \text{with }\ b_2=2.$$

$\Omega_{\mathrm{SVIP}}=\left\{x=\begin{bmatrix}0&t&-t\end{bmatrix}^\top\mid t\in\mathbb{R}:-0.5\le t\le 1\right\}$. The unique solution of VIP $\left(I^{\mathbb{R}^3}-G,\Omega_{\mathrm{SVIP}}\right)$ is $x^*=\begin{bmatrix}0&0&0\end{bmatrix}^\top$. Now, choose $\alpha_n=n^{-0.5}$, $\rho_n=0.75$, $e_n=0.25$, $\lambda=0.4$, and $\tau=0.5$, tolerance $\varepsilon=10^{-3}$ and initial point $x^1=\begin{bmatrix}1&-2&1\end{bmatrix}^\top$, we get

$$x=\begin{bmatrix}1.6354\times10^{-4}&8.4072\times10^{-4}&4.8984\times10^{-10}\end{bmatrix}^\top.$$

This result archived within 31.25×10^{-3} seconds and $n=56$

Next, we used different choices of parameters. Table 1 shown below is the performance with different α_n parameter, $\rho_n=0.75$, $e_n=0.25$, $\lambda=0.4$, and $\tau=0.5$.

Table 1. *Result with different choices of* α_n

ε	$\alpha_n=n^{-0.2}$		$\alpha_n=n^{-0.5}$		$\alpha_n=n^{-0.8}$	
	$\|x-x^*\|$	**Iter.** (n)	$\|x-x^*\|$	**Iter.** (n)	$\|x-x^*\|$	**Iter.** (n)
10^{-3}	0.81×10^{-3}	20	0.99×10^{-3}	56	0.99×10^{-3}	649
10^{-6}	0.99×10^{-6}	43	0.98×10^{-6}	202	10^{-6}	10841
10^{-9}	0.82×10^{-9}	71	10^{-9}	441	10^{-9}	65023

Then we changed the contraction mapping G, with the same choice of parameters, as $\alpha_n = n^{-0.5}$, $\lambda = 0.4$, $e_n = 0.75$, $\rho_n = -\frac{1}{4}\sin\pi n + \frac{1}{2}$ and initial point $x^1 = [1 \quad -2 \quad 1]^\top$. The results are recorded in Table 2.

Table 2. *Result with different contraction mappings G*

ε	$\tau = 0.8$		$\tau = 0.4$		$\tau = 0.2$	
	$\|x - x^*\|$	**Iter.** (n)	$\|x - x^*\|$	**Iter.** (n)	$\|x - x^*\|$	**Iter.** (n)
10^{-3}	0.99×10^{-3}	333	0.91×10^{-3}	39	0.96×10^{-3}	20
10^{-6}	0.99×10^{-6}	1256	0.99×10^{-6}	137	0.95×10^{-6}	72
10^{-9}	10^{-9}	2773	0.97×10^{-9}	301	0.97×10^{-9}	159

4. Conclusion

In this paper, we introduced a self-adaptive iterative algorithm (Algorithm 1) and a strong convergence theorem (Theorem 3.1) for solving the (BVIP) in a real Hilbert spaces without prior knowledge of operator's norms. We consider a numerical example to illustrate the effectiveness of the proposed algorithm.

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