THE EXISTENCE AND UNIQUENESS OF THE INVARIANT MEASURE FOR THE SOLUTION OF SDEs WITH NONLINEAR COEFFICIENTS

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ABSTRACT

The invariant measure is one of the important properties of stochastic differential equations (SDEs). This problem has been well studied for SDEs with regular coefficients. However, there are many open questions in the case of low regular coefficients or irregular coefficients. One of the important questions is that the conditions of the coefficients lead to the existence and uniqueness of the invariant measure. In this paper, we consider SDEs with low regular coefficients. More precisely, this paper considers SDEs with the super-linear, locally Lipschitz continuous coefficients, and coefficients satisfy the contractive condition. The paper shows the existence and uniqueness of the solution of this equation. The author also studies the moment stability of the solution. The main result of the paper shows the existence and uniqueness of the invariant measure of the solution.

SỰ TỒN TẠI VÀ DUY NHẤT PHÂN PHỐI DỪNG CỦA NGHIỆM ĐỐI VỚI PHƯƠNG TRÌNH VI PHÂN NGẪU NHIÊN VỚI HỆ SỐ KHÔNG TUYẾN TÍNH

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TỪ KHÓA

Phương trình vi phân ngẫu nhiên
Liên tục Lipschitz địa phương
Tăng trưởng đa thức
Phân phối dừng
Ổn định theo phân phối

TÓM TẮT

Phân phối dừng là một trong những tính chất quan trọng của phương trình vi phân ngẫu nhiên. Vấn đề này đã được nghiên cứu cho phương trình với hệ số chính quy. Tuy nhiên còn rất nhiều câu hỏi mở trong trường hợp hệ số chính quy yếu hoặc không chính quy. Một trong những câu hỏi quan trọng là điều kiện của hệ số dẫn đến sự tồn tại duy nhất của phân phối dừng. Bài báo này xét phương trình vi phân ngẫu nhiên với hệ số chính quy yếu. Cụ thể hơn, bài báo này xét phương trình vi phân ngẫu nhiên với hệ số tăng trên tuyến tính, liên tục Lipschitz địa phương và thỏa mãn điều kiện co rút. Bài báo chỉ ra sự tồn tại và duy nhất nghiệm của phương trình. Tác giả cũng xét sự ổn định theo moment của nghiệm. Kết quả chính của bài báo chỉ ra sự tồn tại duy nhất của phân phối dừng của nghiệm.

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1. Introduction

Stochastic differential equations arise in many areas of science and engineering: biological, physical, economical,....(see [1]-[3]). So it has attracted much attention recently. Invariant measure is one of essential properties of stochastic differential equations (SDEs), when long time behaviours of SDEs are investigated, such as the persistence for biology and epidemic SDE models (see [1], [2]). Moreover, in many approximation schemes, besides considering the strong order convergence in L^p norm, we also study the order of convergence of the invariant measure of the appoximation solution to the invariant measure of the exact solution. It is known that, SDEs with regular coefficient have been well studied (see [4], [5]). But in the case the coefficients satisfy low regular conditions or irregular conditions, there are many open questions. One of the interesting questions is that: what conditions ensure the existence and uniqueness of the invariant measure of a SDEs. There are some works which deal with this problem (see [6] -[8]). In [6], authors release the global Lipschitz condition on the drift coefficient by assuming the one-sided Lipschitz condition instead, but they still require the global Lipschitz condition on the diffusion coefficient. In [7], the paper considers the existence and uniqueness of the invariant measure for a stochastic differential equation with Markovian switching in the case coefficients satisfy the local Lipschitz condition and the linear growth condition. The open questions is that about the the existence and uniqueness of the invariant measure in the case super- linear growth coefficients.
From this motivation, in this paper, we consider on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a one-dimensional process $Y = (Y_t)_{t\geq 0}$ solution to the following stochastic differential equation

$$Y_t = y_0 + \int_0^t f(Y_s)ds + \int_0^t g(Y_s)dW_s, \tag{1}$$

for $t \geq 0$, where the initial condition $y_0 \in \mathbb{R}$ and $W = (W_t)_{t\geq 0}$ is a one-dimensional standard Brownian motion. The coefficient f and g satisfy the following conditions

C1. There exist positives constant C_1, m such that

$$|f(x) - f(y) \leq C_1(1 + |x|^m + |y|^m)|x - y|,$$

for all $x, y \in \mathbb{R}$.

C2. There exist positive constants C_2 and l such that

$$|g(x) - g(y)| \leq C_2(1 + |x|^l + |y|^l)|x - y|,$$

for all $x, y \in \mathbb{R}$.

C3. There exist constants $\gamma < 0, \eta \in [0, +\infty)$ and $p_0 \in [2, +\infty)$ such that

$$xf(x) + \frac{p_0 - 1}{2} g^2(x) \leq \gamma x^2 + \eta,$$

for any $x \in \mathbb{R}$.

C4. There exists a negative constant λ such that

$$(x-y)(f(x)-f(y))+\frac{1}{2}|g(x)-g(y)|^2 \le \lambda|x-y|^2,$$

for all $x, y \in \mathbb{R}$.

The rest of this paper is organized as follows. In Section 2, we consider the moment stability of the exact solution. The main result shows the existence and uniqueness of the invariant measure of the exact solution in Section 3.

2. Moment estimates

Firstly, note that Conditions **C2** is the special case of Condition **A4** in [9] with $\alpha = 1/2$, and Conditions **C3** is the special case of Condition **A1** in [9]. The conditions **C1, C2** imply that both coefficients f and g are locally Lipschitz continuous and polynomial growth

$$|f(x)| \le C(1+|x|^{m+1}), \quad |g(x)| \le C(1+|x|^{l+1}) \text{ for all } x \in \mathbb{R}.$$

Then, following [9], the Equation (1) has a unique strong solution. More specifically, we can state as following

Proposition 2.1. *Assume that Conditions **C1, C2** hold. Moreover, Conditions **C3** holds for any $p_0 \ge 4m+4$. Then Equation (1) has a unique strong solution.*

Proposition 2.2. *[[9], Proposition 2.3] Assume that $Y = (Y_t)_{t\ge 0}$ is a solution to equation (1) with the initial value y_0. Assume further that the coefficients f, g satisfy condition **C3**, and that g is bounded on every compact subset of $\mathbb{R}$. Then, for any $p \in (0, p_0]$ and $t \ge 0$,*

$$\mathbb{E}\left[|Y_t|^p\right] \le \left|y_0^2 e^{2\gamma t} + \frac{\eta}{\gamma}(e^{2\gamma t}-1)\right|^{p/2}.$$

Moreover, with the condition $\gamma < 0$, there exists a positive constant $C = C_p$ which does not depend on the time t such that

$$\sup_{t\ge 0} \mathbb{E}\left[|Y_t|^p\right] \le C_p.$$

We denote $Y_t^{x_0}$ is the unique global solution of SDE (1) with the initial value x_0. The following Proposition considers the moment of two solutions of Equation (1) with the initial value x_0, y_0, respectively.

Proposition 2.3. *Assume that $Y_t^{x_0}, Y_t^{y_0}$ are two solutions of Equation (1) with the initial value x_0, y_0, respectively. Suppose that Condition **C4** holds. Then*

$$\lim_{t\to\infty} \mathbb{E}\left[|Y_t^{x_0} - Y_t^{y_0}|^2\right] = 0, \textit{uniformly in } x_0, y_0 \in K,$$

for any compact subset $K \in \mathbb{R}$.

Proof: From Equation (1), we have

$$d(Y_t^{x_0} - Y_t^{y_0}) = (f(Y_t^{x_0}) - f(Y_t^{y_0}))dt + (g(Y_t^{x_0}) - g(Y_t^{y_0}))dW_t.$$

Using Ito's formula

$$\begin{aligned}
&e^{-2\lambda t}|Y_t^{x_0} - Y_t^{y_0}|^2 = |x_0 - y_0|^2 + \\
&+ \int_0^t e^{-2\lambda s}\left[-2\lambda|Y_s^{x_0} - Y_s^{y_0}|^2 + 2(Y_s^{x_0} - Y_s^{y_0})(f(Y_s^{x_0}) - f(Y_s^{y_0})) + |g(Y_s^{x_0}) - g(Y_s^{y_0})|^2\right] dt \\
&+ \int_0^t 2e^{-2\lambda s}(Y_s^{x_0} - Y_s^{y_0})(g(Y_s^{x_0}) - g(Y_s^{y_0}))dW_t.
\end{aligned}$$

By using Condition **C4**, we obtain

$$e^{-2\lambda t}|Y_t^{x_0} - Y_t^{y_0}|^2 \leq |x_0 - y_0|^2 + \int_0^t 2e^{-2\lambda s}(Y_s^{x_0} - Y_s^{y_0})(g(Y_s^{x_0}) - g(Y_s^{y_0}))dW_t. \tag{2}$$

By using Proposition 2.1, the stochastic integral in (2) is a square integrable martingale. Then the expectation of this stochastic integral is equal to zero. Therefore, from (2) we get

$$\mathbb{E}[e^{-2\lambda t}|Y_t^{x_0} - Y_t^{y_0}|^2] \leq |x_0 - y_0|^2.$$

It leads to

$$\mathbb{E}[|Y_t^{x_0} - Y_t^{y_0}|^2] \leq e^{2\lambda t}|x_0 - y_0|^2.$$

Let $t \to \infty$, we obtain the desired result.

3. Main results

The main result of this paper shows the existence and uniqueness of the invariant measure for the exact solution of the Equation (1) which is stated as following

Theorem 3.1. *Assum that Conditions **C1- C4** hold. Then the unique solution of Equation* (1) *has a unique invariant measure.*

Proof: By using Theorem 9.1 and Theorem 9.5 in [2], the unique solution of Equation (1) Y_t^x with an initial value x is a time-homogeneous Markov process. Let $\mathbb{P}(t, x,)$ denote the transition probability of the process . Let $\mathcal{P}(\mathbb{R})$ denote all probability measures on $\mathbb{R}$. For any $\mathbb{P}_1, \mathbb{P}_2$ in $\mathcal{P}(\mathbb{R})$, we define a metric d_Θ as following

$$d_\Theta(\mathbb{P}_1, \mathbb{P}_2) = \sup_{\phi\in\Theta} |\int_{\mathbb{R}} \phi(x)\mathbb{P}_1 dx - \int_{\mathbb{R}} \phi(x)\mathbb{P}_2 dx|,$$

where

$$\Theta = \{\phi : \mathbb{R} \to \mathbb{R} : |\phi(x) - \phi(y) \leq |x - y|\}.$$

Given a compact set $K \subset \mathbb{R}$, for any $x, y \in K$, and for any $\phi \in \Theta$, we have

$$|\mathbb{E}[\phi(Y_t^x)] - \mathbb{E}[\phi(Y_t^y)]| \leq \mathbb{E}[|Y_t^x - Y_t^y|] \leq [\mathbb{E}[|Y_t^x - Y_t^y|^2]]^{1/2}.$$

By using Proposition 2.3, we obtain that there exists a $T > 0$ such that

$$|\mathbb{E}[\phi(Y_t^x)] - \mathbb{E}[\phi(Y_t^y)]| \leq \epsilon \text{ for all } t > T.$$

Since ϕ is arbitrary, we have

$$\sup_{\phi \in \Theta} |\mathbb{E}[\phi(Y_t^x)] - \mathbb{E}[\phi(Y_t^y)]| \leq \epsilon \text{ for all } t > T.$$

It leads to

$$d_\Theta(\mathbb{P}(t, x, .), \mathbb{P}(t, y, .)) \leq \epsilon \text{ for all } t > T.$$

Let $t \to \infty$, we get that

$$\lim_{t \to \infty} d_\Theta(\mathbb{P}(t, x, .), \mathbb{P}(t, y, .)) = 0, \tag{3}$$

uniformly in $x, y \in K$.
Moreover, for any $\phi \in \Theta, y_0 \in \mathbb{R}, t, s > 0$, we have

$$\begin{aligned}
&|\mathbb{E}[\phi(Y_{t+s}^{y_0})] - \mathbb{E}[\phi(Y_t^{y_0})]| = |\mathbb{E}[\mathbb{E}[\phi(Y_{t+s}^{y_0})]|\mathcal{F}_s] - \mathbb{E}[\phi(Y_t^{y_0})]| \\
&= |\mathbb{E}[\int_{-\infty}^{+\infty} \phi(Y_t^y)\mathbb{P}(s, y_0, dy)] - \mathbb{E}[\phi(Y_t^{y_0})]| \\
&\leq \int_{-\infty}^{+\infty} |\mathbb{E}[\phi(Y_t^y)] - \mathbb{E}[\phi(Y_t^{y_0})]|\mathbb{P}(s, y_0, dy) \\
&\leq \int_{-N}^{N} |\mathbb{E}[\phi(Y_t^y)] - \mathbb{E}[\phi(Y_t^{y_0})]|\mathbb{P}(s, y_0, dy) + 2\mathbb{P}(s, y_0, K_N^c),
\end{aligned} \tag{4}$$

where $K_N^c = \{y \in \mathbb{R} : |y| > N\}$. By using Proposition 2.2, there exits a positive constant $N > |y_0|$ sufficiently large such that

$$\mathbb{P}(s, y_0, K_N^c) \leq \frac{\epsilon}{4} \quad \text{for all } s > 0. \tag{5}$$

Using Proposition 2.3, there exists $T > 0$ such that

$$\sup_{\phi \in \Theta} \mathbb{E}[\phi(Y_t^y)] - \mathbb{E}[\phi(Y_t^{y_0})] \leq \frac{\epsilon}{2} \quad \forall t \geq T, \forall |y| \leq N. \tag{6}$$

Substituting (5) and (6) in to (4) we get

$$|\mathbb{E}[\phi(Y_{t+s}^{y_0})] - \mathbb{E}[\phi(Y_t^{y_0})]| \leq \epsilon \quad \forall t \geq T, s > 0. \tag{7}$$

Since ϕ is arbitrary then,

$$\sup_{\phi \in \Theta} |\mathbb{E}[\phi(Y_{t+s}^{y_0})] - \mathbb{E}[\phi(Y_t^{y_0})]| \leq \epsilon \quad \forall t \geq T, s > 0.$$

It implies that

$$d_\Theta(\mathbb{P}(t+s, y_0, .); \mathbb{P}(t, y_0, .) \le \epsilon \quad \forall t \ge T, s > 0, y_0 \in \mathbb{R}.$$

It leads to that for any $y_0 \in \mathbb{R}$, $\{\mathbb{P}(t, y_0, .) : t > 0\}$ is a Cauchy sequence in the space $\mathcal{P}(\mathbb{R})$ with metric d_Θ.

Since $\mathbb{R}$ is complete and separable then any sequence $\{\mathbb{P}(t_n, y_0, .)\}$ ($t_n \to \infty$ as $n \to \infty$) has a weak convergent subsequence. Assume its weak limit is an invariant measure $\mu(.)$. Then there exists a positive integer N such that $t_N > T$ and

$$d_\Theta(\mathbb{P}(t_n, y_0, .); \mu(.)) \le \epsilon \quad \forall n \ge N.$$

We have

$$\begin{aligned} d_\Theta(\mathbb{P}(t, y_0, .); \mu(.)) &\le d_\Theta(\mathbb{P}(t_n, y_0, .); \mu(.)) + d_\Theta(\mathbb{P}(t_n, y_0, .); \mathbb{P}(t, y_0, .)) \\ &\le 2\epsilon \quad \forall t \ge T. \end{aligned}$$

Then,

$$\lim_{t\to\infty} d_\Theta(\mathbb{P}(t, y_0, .); \mu(.)) = 0,$$

and the invariant measure $\mu(.)$ is unique. For any $x_0 \in \mathbb{R}$,

$$\lim_{t\to\infty} d_\Theta(\mathbb{P}(t, x_0, .); \mu(.)) \le \lim_{t\to\infty} d_\Theta(\mathbb{P}(t, x_0, .); \mathbb{P}(t, y_0, .)) + \lim_{t\to\infty} d_\Theta(\mathbb{P}(t, y_0, .); \mu(.)) = 0$$

It leads to that $\mu(.)$ is the unique invariant measure of the solution of Equation (1). Therefore, the proof is complete.

4. Conclusion

The main result of this paper is to consider the existence and uniqueness of the invariant measure for the exact solution of SDEs (1) with both super- linear growth coefficients. Another interesting future work is to investigate the convergence rates of the distributions of some numerical methods for SDEs (1).

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