Tracking Control of Wheeled Mobile Robot
With Model Uncertainties and Input Disturbance:
A Novel Approach with Disturbance Estimation and
Arbitrary Convergence Time Controller

AUTHORS

Le Ngoc Quyên¹, Nguyen Hoai Nam¹*, Pham Nhat Tuan Minh²
¹Hanoi University of Science and Technology, ²Hanoi Amsterdam High School for Gifted

ARTICLE INFO

Received: 14/10/2021
Revised: 30/11/2021
Published: 30/11/2021

KEYWORDS

Disturbance estimator
Arbitrary convergence time
Wheeled mobile robots
Nonlinear control
Tracking control

ABSTRACT

Wheeled mobile robots have been widely applied in practice and they have drawn a lot of interests from the research community due to their non-holonomic constraints, nonlinearity and uncertain load. In this work, a novel tracking control approach is proposed for wheeled mobile robots under model uncertainties and input disturbance. The new approach is based on a disturbance estimator and an arbitrary convergence time controller. The model uncertainties and input disturbance will be compensated by the disturbance estimator whereas velocity errors will converge to zero in small prescribed settling time by the arbitrary convergence time controller, which will improve control performance of the closed-loop system. The effectiveness of the proposed method will be verified through numerical simulations.

DOI: https://doi.org/10.34238/jst.5172

* Corresponding author. Email: nam.nguyenhoai@hust.edu.vn

http://jst.tnu.edu.vn
1. Introduction

Automated Guided Vehicles (AGVs) have been widely studied and applied in industry [1]. There are several types of AGVs but the type of three-wheeled mobile robot (WMR) [2] will be considered in this work due to its nonlinearity, underactuation property and nonholonomic constraint, which leads to one of the most difficult control problems for AGVs. Some advanced control methods have been developed for the WMR such as adaptive sliding mode control [3], nonlinear control based on extended state observer [4], adaptive tracking control [5] for the WMR with the center located at the middle of wheels’ axis and model predictive control [6]. These controllers were either complex in implementation or require high computational load.

In this paper, a novel supplemental method to the existing ones is proposed for tracking control of WMR under conditions of model uncertainty and input disturbance. The main contribution of this work is a) to develop a finite time controller for the velocity of WMR and b) to apply a novel disturbance estimation technique for removing effect of uncertainty and disturbance.

The remaining part of this paper is organized as follows. In section 2, a mathematical model of the WMR is briefly given first, then a traditional tracking controller is provided, after that an arbitrary time convergence controller is designed to improve control performance by producing desired velocities for tracking controller as fast as possible, finally a disturbance estimator is developed to suppress the impact of model uncertainty and disturbance. In section 3, numerical simulations are carried out for the WMR and comparison is also made when the disturbance estimator is not applied. Final section will draw some conclusions and provide future work.

2. Main results

2.1. Mathematical model

A schematic diagram of WMRs is shown in Fig. 1, in which \((x_c, y_c)\) are center coordinates of the WMR, \(d\) is the distance between the center and wheel’s axis, \(r\) is the radius of wheels, \(2R\) is the distance between two wheels and \(\theta\) is the WMR’s orientation angle. A model of the WMR [2] can be represented as:

\[
\overset{\cdot}{\overset{\cdot}{\mathbf{q}}} + \overset{\cdot}{\mathbf{V}}(\mathbf{q}) \overset{\cdot}{\mathbf{q}} + \mathbf{F}(\mathbf{q}) + \mathbf{G}(\mathbf{q}) + \tau_d = \mathbf{B}(\mathbf{q}) \tau - \mathbf{A}(\mathbf{q}) \lambda,
\]

where \(\mathbf{q} = [x_c, y_c, \theta]^T\), \(\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{3 \times 3}\) is a symmetric, positive definite inertia matrix, \(\mathbf{V}(\mathbf{q}, \mathbf{\dot{q}}) \in \mathbb{R}^{3 \times 1}\) is the centripetal and Coriolis matrix, \(\mathbf{F}(\mathbf{q}) \in \mathbb{R}^{3 \times 1}\) is the surface friction vector, \(\mathbf{G}(\mathbf{q}) \in \mathbb{R}^{3 \times 1}\) is the gravitational vector, \(\tau_d\) is the unknown disturbance vector, \(\mathbf{B}(\mathbf{q}) \in \mathbb{R}^{3 \times 2}\) is the input transformation matrix, \(\mathbf{A}(\mathbf{q}) \in \mathbb{R}^{1 \times 3}\) is the matrix associated with constraints and \(\lambda \in \mathbb{R}^{1 \times 1}\) is the constraint force vector.

\[\text{Figure 1. A schematic diagram of WMR.}\]

It is assumed that the WMR can only roll, and it does not slip [2]. So, the following equation holds:

http://jst.tnu.edu.vn

Email: jst@tnu.edu.vn
Let $$\dot{y}_c \cos \theta - \dot{x}_c \sin \theta - d \dot{\theta} = 0.$$ \hspace{1cm} (2)

Let

$$S(q) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad v = \begin{bmatrix} v \\ \omega \end{bmatrix}.$$ 

where $$v$$ and $$\omega$$ are linear and angular velocities, respectively. Then,

$$\begin{bmatrix} \dot{x}_c \\ \dot{y}_c \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}.$$ \hspace{1cm} (3)

The following terms are obtained by using the Lagrange method, in which $$G(q) = 0$$ due to assumption of moving on horizontal plane.

$$M(q) = \begin{bmatrix} m & 0 & m \cos \theta \\ 0 & m & -m \sin \theta \\ m \sin \theta & -m \cos \theta & I \end{bmatrix}, \quad V(q, \dot{q}) = \begin{bmatrix} 0 & 0 & m \dot{\theta} \cos \theta \\ 0 & 0 & m \dot{\theta} \sin \theta \\ 0 & 0 & 0 \end{bmatrix}.$$ 

$$A(q)^T = \begin{bmatrix} -\sin \theta & \cos \theta \\ \cos \theta & \cos \theta \\ -d & -\theta \end{bmatrix}, \quad B(q) = \frac{1}{r} \begin{bmatrix} \cos \theta & \cos \theta \\ \sin \theta & \sin \theta \\ R & -R \end{bmatrix}, \quad \lambda = -m(\dot{x}_c \cos \theta + \dot{y}_c \sin \theta) \dot{\theta}.$$ 

Thus,

$$S^T(q)A^T(q) = 0.$$ \hspace{1cm} (4)

Premultiplying both sides of Eq. (1) with $$S^T$$ to have

$$\dot{q} = S \dot{v}.$$ \hspace{1cm} (5)

$$S^TMS\dot{v} + S^T(MS + VS)\dot{v} + S^TF + S^T\tau_d = S^TB \dot{\tau}.$$ \hspace{1cm} (6)

Denote $$\overline{M}(q) = S^TMS, \overline{V}(q, \dot{q}) = S^T(MS + VS), \overline{F}(v) = S^TF, \overline{\tau}_d = S^T\tau_d, \overline{B} = S^TB.$$ Then, Eq. (6) is rewritten as:

$$\overline{M}(q)v + \overline{V}(q, \dot{q})v + \overline{F}(v) + \overline{\tau}_d = \overline{B} \dot{\tau}.$$ \hspace{1cm} (7)

The system (7) will be used to design controllers in next section.

### 2.2. Tracking control

Given reference signal $$q_r(t) = [x_r(t), y_r(t), \theta_r(t)]^T$$. The target is to find forces $$\tau(t)$$ applying to left and right wheels to the WMR such that $$q(t) - q_r(t) \to 0$$ as $$t \to \infty$$. This control problem poses some following issues: 1) the system is under-actuated because there are two inputs $$\tau_1, \tau_2$$ but three output $$x(t), y(t), \theta(t)$$; 2) it is affected by disturbance and model uncertainty; 3) there exists a non-holonomic constraint; and 4) the references are time varying.

A block diagram of the proposed control method is shown in Fig. 2.

The proposed method consists of two control loops where the inner loop including an arbitrary convergence time controller [7] and disturbance estimator [8], and the outer loop is a traditional controller [3].

Let $$v_r(t) = [v_r \quad \omega_r]^T$$ be the reference velocity. According to Eq. (5), it is obtained:

$$\dot{q}_r = S \dot{v}_r.$$ \hspace{1cm} (8)

Define

$$q_e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_r - x \\ y_r - y \\ \theta_r - \theta \end{bmatrix}. \hspace{1cm} (9)$$

Then,

$$\dot{q}_e = \begin{bmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \frac{e_1}{e_2} + \omega \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$ \hspace{1cm} (10)

To achieve that $$q(t) - q_r(t) \to 0$$ as $$t \to \infty$$, the outer loop controller [3] is designed as:
where $k_1, k_2, k_3$ are positive constants. The overall control system is shown in Fig. 2.

Figure 2. Control system diagram.

2.3. Velocity controller design

To force the WMR’s velocities to track the output of the kinematic controller ($\mathbf{v}_c$, $\mathbf{\omega}_c$) as soon as possible, we will design an arbitrary convergence time controller using the novel control technique [7].

It is assumed that all the uncertainties and disturbances are zero. These unknown terms will be compensated by their estimated values from a disturbance estimator. Define $\mathbf{u} = [u_1 \ u_2]^T$ as new auxiliary input and design a feedback linearization controller as

$$
\mathbf{\tau} = \mathbf{f}(\mathbf{q}, \mathbf{q}, \dot{\mathbf{v}}, \mathbf{u}) = \mathbf{B}^{-1} \mathbf{q} \begin{bmatrix} \mathbf{M}(\mathbf{q}) \mathbf{u} + \mathbf{V}(\mathbf{q}, \mathbf{q}) \mathbf{v} \end{bmatrix}.
$$

Then, the system (7) becomes

$$
\ddot{\mathbf{v}} = \mathbf{u} - \mathbf{v}.
$$

Denote velocity error vector as $\mathbf{e}_v = \begin{bmatrix} \mathbf{e}_v \ \mathbf{e}_w \end{bmatrix} = \mathbf{v}_c - \mathbf{v} = \begin{bmatrix} \mathbf{v}_c - \mathbf{v} \\
\mathbf{\omega}_c - \mathbf{\omega} \end{bmatrix}$. 

Theorem 1.

With following control law

$$
\mathbf{u} = \mathbf{C} + \mathbf{v}_c,
$$

where

$$
\mathbf{C} = \begin{bmatrix}
n_1(e^{-e_v} - 1) \\
e^{-e_v}(t_f - t) \\
n_2(e^{-e_\omega} - 1) \\
e^{-e_\omega}(t_f - t)
\end{bmatrix}
$$

for $t < t_f$

$$
\begin{cases}
0, & \text{for } t \geq t_f
\end{cases}
$$

$\mathbf{e}_v \to \mathbf{0}$ after an arbitrary small time $t_f > 0$.

Proof:

Choose a Lyapunov candidate function as

$$
V_d = ||\mathbf{e}_v||^2
$$

Clearly, $V_d \geq 0$ and $V_d = 0$ if and only if $\mathbf{e}_v = \mathbf{0}$. Time derivative of $V_d$ is

$$
\dot{V}_d = 2\mathbf{e}_v^T \dot{\mathbf{e}}_v = 2\mathbf{e}_v^T (\dot{\mathbf{v}}_c - \dot{\mathbf{v}}) = 2\mathbf{e}_v^T (\dot{\mathbf{v}}_c - \mathbf{u}) = -2\mathbf{e}_v^T \mathbf{C}
$$

Consider a function $f_1(x) = \begin{cases}
x(e^{-x} - 1) & \text{if } x \geq 0 \\
0 & \text{if } x < 0
\end{cases} = x(1 - e^{-x}).$ One has

$$
f_1(x) - f_1(-|x|) = \begin{cases}
|x(1 - e^x) + x(1 - e^{-x}) = x(2 - e^x - e^{-x}), & \text{when } x \geq 0 \\
0 & \text{when } x < 0
\end{cases}
$$

We have $2 - e^x - e^{-x} \leq 0, \forall x$, but $x > 0 \Rightarrow f_1(x) - f_1(|x|) < 0$. Thus,

$$
f_1(x) \leq f_1(-|x|), \forall x
$$

(i) As $t < t_f$
\[
\dot{V}_d = -2 \left[ -\eta_1 e_v (e^{-e_v} - 1) \frac{e^{-e_v} (t_f - t)}{e^{-e_v} (t_f - t)} - \eta_2 e_\omega (e^{-e_\omega} - 1) \frac{e^{-e_\omega} (t_f - t)}{e^{-e_\omega} (t_f - t)} \right]
\]

\[
= 2 \left[ \eta_1 e_v (e^{-e_v} - 1) \frac{e^{-e_v} (t_f - t)}{e^{-e_v} (t_f - t)} + \eta_2 e_\omega (e^{-e_\omega} - 1) \frac{e^{-e_\omega} (t_f - t)}{e^{-e_\omega} (t_f - t)} \right]
\]  

From (19), one gets:

\[
\dot{V}_d \leq -2 \left[ \eta_1 |e_v| (|e_v| - 1) \frac{e^{|e_v|} (t_f - t)}{e^{|e_v|} (t_f - t)} + \eta_2 |e_\omega| (|e_\omega| - 1) \frac{e^{|e_\omega|} (t_f - t)}{e^{|e_\omega|} (t_f - t)} \right]
\]  

Let \( f_2(x) = \frac{x(e^x - 1)}{e^x} = x(1 - e^{-x}) \geq 0, \forall x \). Its derivative is

\[
\hat{f}_2(x) = 1 - e^{-x} + xe^{-x}
\]

\[
\hat{f}_2(x) > 0, \text{ when } x > 0
\]

\[
\hat{f}_2(x) < 0, \text{ when } x < 0
\]

\[
\hat{f}_2(x) = 0, \text{ when } x = 0
\]

From (21) and the inequality \( f_2(x) \geq 0, \forall x \), we obtain

\[
\dot{V}_d \leq -2 \eta_1 |e_v| (|e_v| - 1) \frac{e^{|e_v|} (t_f - t)}{e^{|e_v|} (t_f - t)}
\]

\[
\dot{V}_d \leq -2 \eta_2 |e_\omega| (|e_\omega| - 1) \frac{e^{|e_\omega|} (t_f - t)}{e^{|e_\omega|} (t_f - t)}
\]

Obviously, \( V_d = e_v^2 + e_\omega^2 \leq 2 \max(|e_v|, |e_\omega|) \). This implies that

\[
\frac{\dot{V}_d}{2} \leq \max(|e_v|, |e_\omega|)
\]

so \( \sqrt{\frac{\dot{V}_d}{2}} \leq |e_v| \) or \( \sqrt{\frac{\dot{V}_d}{2}} \leq |e_\omega| \). Without loss of generality, it can be assumed that \( |e_v| \geq |e_\omega| \), thus \( \sqrt{\frac{\dot{V}_d}{2}} \leq |e_v| \). Combine (22) with (23) to get

\[
\dot{V}_d \leq -2 \left[ \eta_1 \frac{\sqrt{\dot{V}_d}}{2} (\sqrt{\dot{V}_d} - 1) \right]
\]

Denote \( \xi = \sqrt{\frac{\dot{V}_d}{2}} \), so \( \dot{\xi} = \frac{1}{2} \frac{\dot{V}_d}{\sqrt{\dot{V}_d}} = \frac{1}{4} \frac{\dot{\xi}}{\xi} \cdot \frac{\dot{V}_d}{\sqrt{\dot{V}_d}} \). Substitute this into Eq. (25) to have

\[
4\xi \dot{\xi} \leq -2 \eta_1 \frac{\xi (\xi - 1)}{e^\xi (t_f - t)} \Rightarrow \dot{\xi} \leq - \eta_1 \frac{\xi (\xi - 1)}{e^\xi (t_f - t)}
\]

where \( \eta_1 = \frac{\eta_1}{2} \).

According to [7], we have \( \xi = 0, \forall t \geq t_f \), so \( V_d = 0, \forall t \geq t_f \) or \( e_v = 0, \forall t \geq t_f \). Similarly, it can be proved for the case \( |e_\omega| \geq |e_v| \). Note that \( \eta_1 \geq 1 \), it means \( \eta_1 \geq 2 \) and \( \eta_2 \geq 2 \).

(ii) As \( t \geq t_f \)

We have \( u = \dot{v}_c \), so \( \dot{v} = \dot{v}_c \), thus \( v(t) - v(t_f) = v_c(t) - v_c(t_f) \). but \( v(t_f) = v_c(t_f) \). This implies \( v(t) = v_c(t) \).

2.4. Disturbance estimator

In this section, a disturbance estimator [8] will be applied for the system (7) with model uncertainty and input disturbance. This disturbance estimator is utilized for the following system:

\[
\dot{x} = \Phi(x, \delta)x + H(x, \delta)(u + d)
\]

A reference model with sampling time \( \delta \) is given as:

\[
z_k = \Phi(z_{k-1}) + H_k(u - d_{k-1}).
\]

Then, the input disturbance will be estimated as follows
\[
\dot{d}_k = [(H_k^T H_k)^{-1} (H_k^T)^T (x_k - z_k - \Phi_k x_{k-1} + \Phi_k z_{k-1})],
\]

where
\[
\Phi_k = I + \delta \Phi(x_{k-1}, t_k), \\
H_k = \delta H(x_{k-1}, t_k), \\
\Phi_k^T = I + \delta \Phi^T(x_{k-1}, t_k).
\]

To compensate the model uncertainty and input disturbance of the system (7), it is rewritten as
\[
\left(M(q) + \Delta M(q)\right)\dot{v} + \left(V(q, \dot{q}) + \Delta V(q, \dot{q})\right)v = B(\tau + \Delta \tau),
\]

with \(\Delta M(q)\), \(\Delta V(q, \dot{q})\) and \(\Delta \tau\) represent the model uncertainty and input disturbance, respectively. Denote \(\Phi(v, t) = -M^{-1}V\), \(H(v, t) = M^{-1}B\) and \(d = \Delta \tau - B^{-1}(\Delta M \dot{v} + \Delta V \ddot{v} + F(v))\), then the system is rewritten as:
\[
\dot{v} = \Phi(v, t)v + H(v, t)(\tau + d).
\]

Finally, the disturbance estimator (29) will be applied for the system (32). In combination with the controllers (12) and (14), the real input to the system (31) is
\[
\tau - \dot{d}_k = f(q, \dot{q}, v, u) = B^{-1}(q)[M(q)(C + \dot{v}_c) + V(q, \dot{q})\dot{v} - \dot{d}_k],
\]

for all time such that \(k\delta \leq t < (k + 1)\delta\) with \(k = 0, 1, 2, \ldots\)

In the next section, the arbitrary convergence time controller (14) in combination with disturbance estimator (29), which is also (33), will be verified through numerical simulations.

3. Numerical simulations

Nominal values of the WMR’s parameters are given as follows: \(m = 4\ (kg)\), \(I = 2.5\ (kg.m^2)\), \(R = 0.15\ (m)\), \(r = 0.03\ (m)\), and \(d = 0.15\ (m)\). These values are used for computing control signal and estimating disturbance. To create the model’s uncertainty for the WMR, the parameters for simulating the plant are increased as follows: \(m = 6\ (kg)\), \(I = 5\ (kg.m^2)\), \(R = 0.18\ (m)\), \(r = 0.036\ (m)\) and \(d = 0.18\ (m)\).

A square reference trajectory will be used with following desired longitudinal and angular velocities.

\[
\begin{align*}
\nu_r &= \begin{cases} 
0.785 \ (m/s), & 0 \leq t \leq 3.146 + 10n \\
1 \ (m/s), & t > (3.146 + 10n) 
\end{cases}, \\
\omega_r &= \begin{cases} 
0.5 \ (rad/s), & 0 \leq t \leq 3.146 + 10n \\
0 \ (rad/s), & t > (3.146 + 10n) 
\end{cases},
\end{align*}
\]

with \(n = 0, 1, 2, 3\). The initial position of the reference trajectory is set at \(q_r(0) = [x_r(0), y_r(0), \theta_r(0)]^T = [0, 0, 0]^T\) and that of the WMR is given as \(q(0) = [x(0), y(0), \theta(0)]^T = [2, 2, \pi]^T\).

The unknown input disturbance is given for simulation as \(\Delta \tau = \begin{bmatrix} 2 \sin(10t) + 2 \cos(5t) \\
1 \sin(5t) + 3 \cos(10t) \end{bmatrix}\) and \(t_f = 5\ (s)\).

Fig. 3 shows numerical simulation results. The proposed method provided very small tracking control errors in comparison with the case that the new method without disturbance estimator was used.

Some different values of the desired time \(t_f\) were also used for simulation. It can draw a conclusion that as \(t_f\) is decreased the control signal \(\tau\) tends to be bigger for the starting time while the tracking errors are similar.

4. Conclusion and future work

This work proposed a tracking controller for WMRs with model uncertainty and input disturbance, in which an arbitrary convergence time controller for the velocity control loop was associated with a disturbance estimator to eliminate the effect of model uncertainty and input
disturbance and force the velocity errors to converge to zero in arbitrary finite time. The numerical simulation proved the effective of the proposed method.

Future works focus on stability analysis of the overall system and implement on real WMRs to assess the proposed method on practical WMRs.

![Comparison of the proposed method to the case without disturbance estimator.](image)

(a) WMR’s trajectories, (b) control errors for $x$, (c) control errors for $y$, (d) control errors for orientation angle $\theta$.

REFERENCES


