MODEL ORDER REDUCTION OF ELECTRICAL CIRCUITS: A COMPARATIVE STUDY OF POSITIVE-REAL BALANCED TRUNCATION AND ITERATIVE RATIONAL KRYLOV

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ABSTRACT

Model Order Reduction (MOR) is for simplifying complex electrical, electronic circuit models while preserving essential system characteristics. This study compares two MOR algorithms: Positive-Real Balanced Truncation (PRBT) and the Iterative Rational Krylov Algorithm (IRKA). The PRBT approach involves the determination of controller and observer Gramians through the solution of Riccati equations, enabling the preservation of stability and passivity properties. The IRKA employs an iterative of interpolation points to converge towards an optimal reduced-order approximation. The performance of these algorithms is evaluated through the implementation of an RLC circuit model in MATLAB software. The results encompass error analysis, comparisons of impulse response, magnitude response, and phase response according to frequency between the original and reduced-order systems. The strengths and limitations of each method, providing into their applicability involving model order reduction of electrical circuits. By considering factors such as accuracy, computational complexity, and preservation of system properties, we can make decisions regarding the selection of MOR algorithms for specific circuit modeling tasks. This study contributes to the ongoing advancement of MOR techniques in electrical engineering, facilitating the development of efficient and accurate application circuit models.

GIẢM BẬC MÔ HÌNH CHO MẠCH ĐIỆN: MỘT NGHIÊN CỨU SO SÁNH VỀ CẮT NGẮN CÂN BẰNG THỰC DƯƠNG VÀ LẶP KRYLOV HỮU TỈ

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TỪ KHÓA

Giảm bậc mô hình
Cắt ngắn cân bằng thực dương
Thuật toán lặp Krylov hữu tỉ
Mạch điện
Mô hình mạch

TÓM TẮT

Giảm bậc mô hình (MOR) nhằm đơn giản hóa các mô hình mạch điện, điện tử phức tạp trong khi vẫn duy trì các đặc tính thiết yếu của hệ thống. Nghiên cứu này so sánh hai thuật toán MOR: Cắt ngắn cân bằng dương-thực (PRBT) và Thuật toán lặp Krylov hữu tỉ (IRKA). Cách tiếp cận PRBT liên quan đến việc xác định Gramian điều khiển và quan sát thông qua việc giải các phương trình Riccati, cho phép duy trì tính ổn định và thụ động. IRKA sử dụng phép lặp các điểm nội suy để hội tụ hướng tới xấp xỉ giảm bậc tối ưu. Hiệu suất của các thuật toán này được đánh giá thông qua việc triển khai mô hình mạch RLC trên MATLAB. Các kết quả bao gồm phân tích sai số, so sánh đáp ứng xung, đáp ứng biên độ và đáp ứng pha theo tần số giữa hệ gốc và hệ bậc giảm. Các điểm mạnh và hạn chế của từng phương pháp, mang lại khả năng ứng dụng liên quan đến việc giảm bậc mô hình mạch điện. Bằng cách xem xét các yếu tố như độ chính xác, độ phức tạp tính toán và bảo toàn các thuộc tính của hệ thống, ta có thể đưa ra quyết định về việc lựa chọn thuật toán MOR cho các nhiệm vụ mô hình hóa mạch cụ thể. Nghiên cứu này góp phần thúc đẩy sự tiến bộ không ngừng của MOR trong kỹ thuật điện, tạo điều kiện phát triển các mô hình mạch ứng dụng hiệu quả và chính xác.

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1. Introduction

In many scenarios, such as simulating electrical circuits, modeling objects, system identification, or time-dependent control problems, the dimensionality of the space can be quite large, while the number of inputs and outputs is much smaller. During implementation, high-order systems can lead to computationally infeasible tasks due to hardware limitations, memory constraints, time requirements, reliability concerns, and other restrictions. One approach to address this challenge is through Model Order Reduction (MOR). The goal of MOR is to generate a lower-dimensional system that preserves similar operational characteristics to the original system while simplifying storage requirements, computation time, deployment costs, and other factors. One of the prominent techniques employed for model order reduction in the realm of electrical engineering and electronics is Positive-real balanced truncation (PRBT) alongside the Iterative Rational Krylov Algorithm (IRKA).

PRBT involves the determination of controller and observer Gramians through the solution of two Riccati equations. These Gramians serve as metrics for the system's energy, providing insights into the controllability and observability of individual state variables. By selectively discarding eigenvalues with minimal energy contributions, a lower-order model is derived. Notably, PRBT preserves the inherent passive properties of the original system. For an in-depth exploration of the PRBT algorithm and its applications, pertinent literature can be referenced in [1] - [5].

On the other hand, IRKA employs initial interpolation data, which may be chosen randomly or derived from a simplified reduced-order model. However, it should be noted that IRKA does not ensure the stability of intermediary reduced-order models, even up to the desired reduced order. This instability arises when the initial interpolation data significantly deviates from the optimal data. For further insights into the IRKA algorithm and its applications, relevant studies can be consulted in [6] - [10].

To evaluate the capabilities of PRBT and IRKA, we conducted implementations of these two algorithms using Matlab for model order reduction of electrical and electronic circuit models as presented in [11]. We compared their performance across several key aspects relevant to the model order reduction problem, including error metrics, computational costs, and the preservation of fundamental physical properties of the circuits, namely stability, and passivity.

The objective of our study is to assess the effectiveness and efficiency of PRBT and IRKA in reducing the complexity of circuit models while retaining essential system characteristics. By systematically analyzing and comparing the outcomes of these algorithms, we aim to provide insights into their applicability and suitability for practical engineering applications.

2. Materials and Methods

2.1. Positive-Real Balanced Truncation (PRBT) algorithm

The algorithm for Positive-real balanced truncation (PRBT), as outlined in [1] - [5], is implemented according to the following sequence:

Input: An electrical or electronic circuit modeled using the Modified Nodal Analysis (MNA) method is taking the form of a stable, passive, minimum-phase linear time-invariant system represented by state-space matrices (**A**, **B**, **C**, **D**) of order *n*, with $\mathbf{A} \in \mathbf{R}^{nxn}, \mathbf{B} \in \mathbf{R}^{nxp}, \mathbf{C} \in \mathbf{R}^{pxn}, \mathbf{D} \in \mathbf{R}^{pxp}, \mathbf{D}^{\mathrm{T}} + \mathbf{D} \geq 0$; $G(s) + G(\bar{s}) \geq 0$.

- Step 1: Solve two Riccati equations (1) and (2) to determine the controller Gramian $\mathbf{P_c}$ and observer Gramian $\mathbf{P_o}$ of the original system. $\mathbf{P_c}$ and $\mathbf{P_o}$ are symmetric, positive-definite matrices.

$$\mathbf{A}^{\mathrm{T}}\mathbf{P_c} + \mathbf{P_c}\mathbf{A} + \left(\mathbf{P_c}\mathbf{B} - \mathbf{C}^{\cdot}\right)\left(\mathbf{D} + \mathbf{D}^{\cdot}\right)^{-1}\left(\mathbf{B}^{\cdot}\mathbf{P_c} - \mathbf{C}\right) = \mathbf{0} \quad (1)$$

$$\mathbf{AP_o} + \mathbf{P_oA^{\bullet}} + \left(\mathbf{P_oC^{\bullet}} - \mathbf{B}\right)\left(\mathbf{D} + \mathbf{D^{\bullet}}\right)^{-1}\left(\mathbf{CP_o} - \mathbf{B^{\bullet}}\right) = \mathbf{0} \quad (2)$$

- Step 2: Perform Cholesky decomposition on $\mathbf{P_c}$ and $\mathbf{P_o}$ to obtain (3) and (4). Here, **S** and **R** represent the Cholesky factors, forming lower triangular and invertible matrices.

$$\mathbf{P_c} = \mathbf{SS}^{\mathrm{T}} \quad (3)$$

$$\mathbf{P_o} = \mathbf{RR}^{\mathrm{T}} \quad (4)$$

- Step 3: Analyze the singular values as in (5).

$$\mathbf{R}^{\mathrm{T}}\mathbf{S} = \mathbf{UJV} \quad (5)$$

- Step 4: Compute the balancing transformation matrices **T** and its inverse using (6) and (7).

$$\mathbf{T} = \mathbf{SVJ}^{1/2} \quad (6)$$

$$\mathbf{T}^{-1} = \mathbf{J}^{1/2}\mathbf{U}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} \quad (7)$$

- Step 5: Transform the original system *G(s)* into a balanced realization as in (8).

$$\begin{gathered}\left[\mathbf{T}^{-1}\mathbf{AT}, \mathbf{T}^{-1}\mathbf{B}, \mathbf{CT}, \mathbf{D}\right] = \left[\begin{bmatrix}\mathbf{A_{11}} & \mathbf{A_{12}} \\ \mathbf{A_{21}} & \mathbf{A_{22}}\end{bmatrix}, \begin{bmatrix}\mathbf{B_1} \\ \mathbf{B_2}\end{bmatrix}, \begin{bmatrix}\mathbf{C_1} & \mathbf{C_2}\end{bmatrix}, \mathbf{D}\right] \\ \mathbf{A}_{11} \in \mathbf{R}^{r\times r}, \mathbf{A}_{12} \in \mathbf{R}^{r\times(n-r)}, \mathbf{A}_{21} \in \mathbf{R}^{(n-r)\times r}, \mathbf{A}_{22} \in \mathbf{R}^{(n-r)\times(n-r)}; \\ \mathbf{B}_1 \in \mathbf{R}^{r\times p}, \mathbf{B}_2 \in \mathbf{R}^{(n-r)\times p}; \mathbf{C}_1 \in \mathbf{R}^{p\times r}, \mathbf{C}_2 \in \mathbf{R}^{p\times(n-r)}; \mathbf{D} \in \mathbf{R}^{pxp}\end{gathered} \quad (8)$$

- Step 6: Choose the desired reduced order *r* (*0 < r < n*).

Output: A stable reduced-order approximation G_PRBT described by state-space matrices ($\mathbf{A_{11}}$, $\mathbf{B_1}$, $\mathbf{C_1}$, **D**) of order *r*.

This algorithmic framework facilitates the systematic reduction of the original high-order system while preserving its stability and passivity. It offers a structured approach to achieve model simplification without compromising essential system characteristics, thereby enabling efficient analysis and design in engineering applications.

2.2. IRKA Iterative Rational Krylov Algorithm (IRKA)

The Iterative Rational Krylov Algorithm (IRKA) [6] - [10] is described as follows:

Input: The electrical or electronic circuit is modeled using the Modified Nodal Analysis (MNA) method, resulting in a stable, passive, minimum-phase linear time-invariant system represented by state-space matrices **A**, **B**, **C**, **D**) of order *n*, with the condition for convergence of the IRKA algorithm is that the system satisfies $\mathbf{A} = \mathbf{A}^T, \mathbf{B} = \mathbf{C}^T$.

- Step 1: Initialize the initial interpolation data: $\zeta_i \, (i = 1, \ldots, r)$

Repeat the following two steps until convergence, i.e., until the eigenvalues (poles) of the system in the subsequent interpolation iteration approximate those of the previous interpolation iteration (remain unchanged). At each iteration, IRKA performs Hermite interpolation to approximate the transfer function of the original system. IRKA iteratively refines the interpolation points to optimize the approximation (which correspond to eigenvalues or poles of the intermediate reduced-order system). It starts with r arbitrary interpolation points, and at each iteration, it enforces the necessary optimality conditions according to the H_2 norm.

- Step 2: Construct the model as in (9), (10), and (11).

$$\begin{gathered}\left[\tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}}, \tilde{\mathbf{D}}\right] := \left[\mathbf{W}^{\mathrm{T}}\mathbf{AK}, \mathbf{W}^{\mathrm{T}}\mathbf{B}, \mathbf{CK}, \mathbf{D}\right] \\ \mathbf{W}, \mathbf{K} \in \mathrm{C}^{n\times r}; \mathbf{W}^{\mathrm{H}}\mathbf{K} = \mathbf{I}_r;\end{gathered} \quad (9)$$

$$\mathrm{Im}\,\mathbf{K} = \mathrm{Im}\left[\left(-\mu_1\mathbf{I}_n - \mathbf{A}\right)^{-1}\mathbf{B} \quad \ldots \quad \left(-\mu_r\mathbf{I}_n - \mathbf{A}\right)^{-1}\mathbf{B}\right] \quad (10)$$

$$\operatorname{Im}\mathbf{W} = \operatorname{Im}\left[\left(-\mu_1\mathbf{I}_n - \mathbf{A}\right)^{-T}\mathbf{C} \quad \ldots \quad \left(-\mu_r\mathbf{I}_n - \mathbf{A}\right)^{-T}\mathbf{C}\right] \tag{11}$$

- Step 3: Compute new interpolation data as in (12), (13).

$$\mathbf{X}^{-1}\tilde{\mathbf{A}}\mathbf{X} = diag\left(\zeta_1,\ldots,\zeta_r\right)$$
$$\mathbf{X} \in \mathrm{C}^{r\times r}; \tag{12}$$

$$\mathbf{B} := \mathbf{X}^{-1}\tilde{\mathbf{B}},\ \mathbf{C} := \tilde{\mathbf{C}}\mathbf{X},\ \tilde{\mathbf{D}} := \mathbf{D} \tag{13}$$

Output: A reduced-order approximation G_IRKA is described by state-space matrices ($\mathbf{A_r}$, $\mathbf{B_r}$, $\mathbf{C_r}$, $\mathbf{D_r}$) of order *r*.

The IRKA algorithm iteratively refines the interpolation points to converge to an optimal reduced-order approximation of the original system. By imposing the necessary optimality conditions at each iteration, it systematically improves the accuracy of the reduced-order model while ensuring convergence towards the desired system characteristics. This iterative approach facilitates efficient model reduction for complex electrical and electronic circuits, enabling streamlined analysis and design processes in engineering applications.

3. Results and Discussion

Considering an RLC circuit configured as depicted in Figure 1, we employ the Modified Nodal Analysis (MNA) method to model this circuit in the form of state matrices. The circuit comprises $k = 5$ nodes and order $n = 9$, wherein capacitance values $C = 1F$, inductance $L = 1H$, and resistances $R_1 = 0.5$ *Ohm* and $R_2 = 0.2$ *Ohm* are pre-determined. The state variables include voltages across capacitors and currents through inductors. The input voltage is denoted as u, and the output current is represented as y.

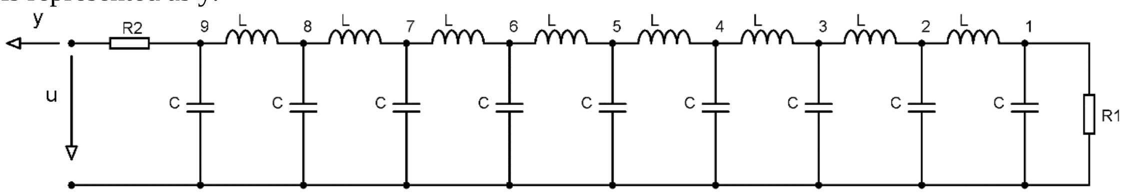

Figure 1. *RLC ladder network*

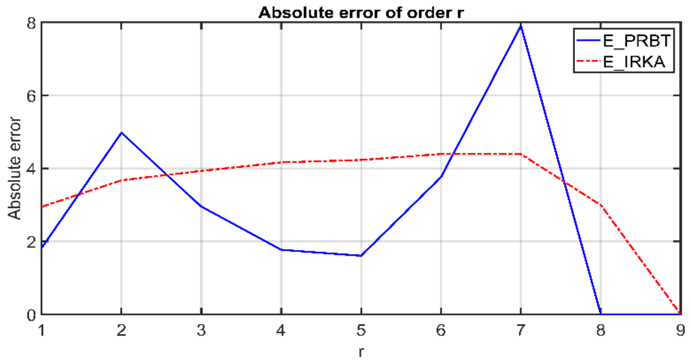

Figure 2. *The absolute error corresponding to the system is reduced in order*

By implementing two algorithms, PRBT and IRKA, in the Matlab environment, we have reduced the order of the system from 9th order to 1st order. The resulting outcomes encompass the absolute error chart corresponding to order r as illustrated in Figure 2 and detailed numerical data as presented in Table 1.

For the case where the order is reduced to $r = 3$, to evaluate the performance of the reduced-order system, we have generated step responses as depicted in Figure 3 and analyzed the magnitude and phase responses with respect to frequency as shown in Figures 4 and 5.

Table 1. *Comparison of Reduction Methods (Absolute error values)*

Reduced Order (r)	PRBT Error	IRKA Error
1	1.814466	2.946739
2	4.974953	3.677551
3	2.954425	4.076892
4	1.769933	4.172695
5	1.605933	4.182282
6	3.762199	4.393198
7	7.905719	4.409214
8	0.000028	2.992839

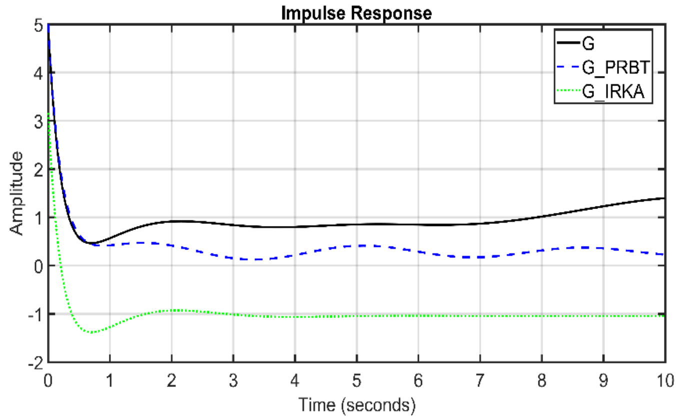

Figure 3. *Impulse response between G, G_PRBT, and G_IRKA*

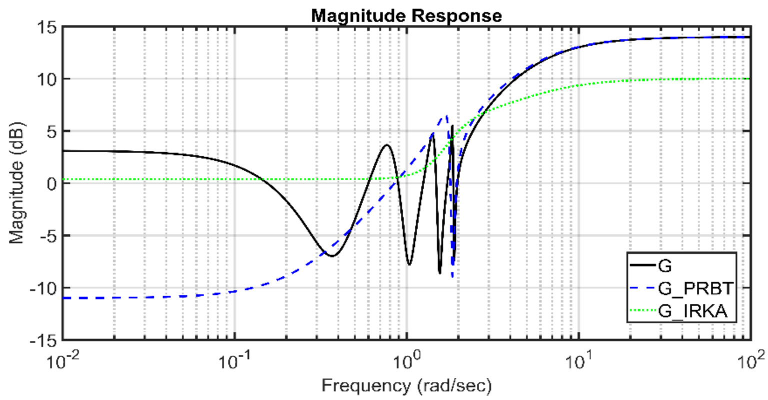

Figure 4. *Magnitude response according to frequency between G, G_PRBT, and G_IRKA*

Upon examining the step response plots among the original system (G), the reduced-order system using PRBT (G_PRBT), and the reduced-order system using IRKA (G_IRKA) as shown in Figure 3, we observe that the response of G_IRKA deviates significantly from G. However, within a time span of less than 1 second, G_PRBT matches the original system, and for larger time spans, it exhibits discrepancies compared to G but with smaller deviations than G_IRKA.

Analysis from Figures 4 and 5 reveals that across the entire frequency range, the magnitude and phase responses of G_IRKA are entirely dissimilar to G. Meanwhile, for frequencies lower than 2 Rad/s, G_PRBT also does not coincide with G, but for frequencies higher than 2 rad/s, G_PRBT aligns with the original system. Therefore, it can be concluded that PRBT can be employed to reduce the system to the 3rd order and substitute for the original system in scenarios operating within this frequency range.

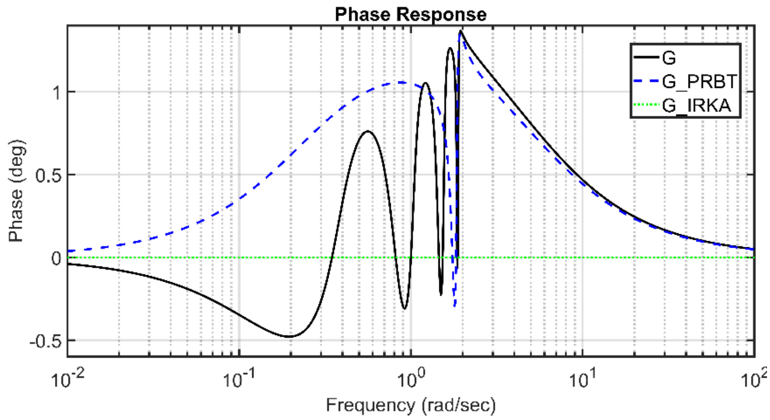

Figure 5. *Phase response according to frequency between G, G_PRBT, and G_IRKA*

4. Conclusion

This study presented a comparative analysis of two prominent model order reduction techniques, Positive-Real Balanced Truncation (PRBT) and the Iterative Rational Krylov Algorithm (IRKA), applied to electrical circuit models. The implementations were performed using MATLAB, with an RLC circuit serving as the test case.

The results demonstrated that PRBT exhibits superior performance in preserving critical system characteristics, such as stability and passivity, compared to IRKA. Notably, the PRBT-reduced system closely matched the original system's step response within a specific time window and aligned with the magnitude and phase responses for frequencies above a certain threshold.

In contrast, IRKA exhibited significant deviations from the original system's behavior across various analyses, highlighting its limitations in accurately capturing the system's dynamics. While IRKA iteratively refines the interpolation points, convergence towards an optimal reduced-order approximation is not guaranteed, potentially leading to inaccurate representations.

The findings underscore the importance of carefully selecting the appropriate model order reduction technique based on the specific requirements and constraints of the application. For electrical circuit modeling, where preserving stability and passivity is crucial, PRBT emerges as a more suitable choice compared to IRKA.

Future research could explore hybrid approaches that combine the strengths of both algorithms or investigate alternative MOR techniques tailored to specific circuit topologies or operating conditions. Additionally, extending the analysis to larger-scale and more complex circuit models would further validate the findings and provide valuable insights for practical engineering applications.

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