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A SELF-ADAPTIVE ITERATIVE METHOD FOR APPROXIMATING THE SOLU- TION OF THE SPLIT FEASIBILITY PROBLEM IN HILBERT SPACES

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ABSTRACT

Since its introduction in 1994, the split feasibility problem has found numerous practical applications in various fields, such as digital engineering and medicine. In this paper, we propose a new iterative method for approximating the solution of the split feasibility problem in real Hilbert spaces. This is achieved by solving a class of variational inequality problems over the solution set of the split feasibility problem. Our algorithm uses inertia techniques to enhance its convergence speed and applies a self-adaptive step size criterion to remove the need to know the norm of transformation operators. By incorporating convex combinations into the algorithm, our method ensures the strong convergence of the generated iterative sequence. This result is proven using mathematical lemmas and theorems under suitable assumptions imposed on the parameters. Finally, a numerical experiment in infinite-dimensional spaces is given to illustrate the effectiveness of our method and compare it with the related ones.

MỘT PHƯƠNG PHÁP LẶP TỰ THÍCH NGHI XẤP XỈ NGHIỆM BÀI TOÁN CHẤP NHẬN TÁCH TRONG KHÔNG GIAN HILBERT

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TỪ KHÓA

Bài toán chấp nhận tách
Bất đẳng thức biến phân
Không gian Hilbert
Ánh xạ không giãn
Điểm bất động

TÓM TẮT

Từ khi được giới thiệu vào năm 1994, bài toán chấp nhận tách đã có những ứng dụng đáng kể trong các lĩnh vực như kỹ thuật số và y học. Trong nghiên cứu này, chúng tôi đề xuất một phương pháp lặp mới để xấp xỉ nghiệm của bài toán chấp nhận tách trong không gian Hilbert thực. Điều này được thực hiện thông qua việc giải một lớp bài toán bất đẳng thức biến phân trên tập nghiệm của bài toán chấp nhận tách. Thuật toán của chúng tôi sử dụng kỹ thuật quán tính để tăng tốc độ hội tụ và áp dụng một tiêu chuẩn cỡ bước tự thích nghi để loại bỏ yêu cầu xác định chuẩn của toán tử chuyển. Tích hợp các tổ hợp lồi vào trong thuật toán, phương pháp của chúng tôi đảm bảo sự hội tụ mạnh của dãy lặp được sinh ra. Kết quả này được chứng minh bằng các mệnh đề và định lý toán học dưới một số giả thiết thích hợp được áp vào các tham số. Cuối cùng, một ví dụ số trong không gian vô hạn chiều được xây dựng để minh họa cho sự hiệu quả của phương pháp và so sánh nó với một số phương pháp liên quan.

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1. Introduction

Let H_0 and H_1 be real Hilbert spaces with nonempty closed convex subsets C and Q, respectively, and $F : H_0 \to H_1$ be a bounded linear mapping. The *Split Feasibility Problem*, proposed by Censor and Elfving [1] in 1994, is stated as follows:

$$\text{Find } u \in C \text{ such that } Fu \in Q. \tag{SFP}$$

Several recent studies [2] - [5] have shown that the (SFP) has numerous practical applications, such as image recovery, signal processing, intensity-modulated radiation therapy and gene regulatory network inference. The most influential method for solving (SFP) is CQ-algorithm presented by Byrne [6] and considered in Hilbert spaces with a weak convergence result by Xu [7]. It starts with an arbitrary initial guess $x^0 \in H_0$ and generates x^{k+1} according to the recursion process

$$x^{k+1} = P_C(I^{H_0} - \gamma_k F^*(I^{H_1} - P_Q)F)x^k, \quad k \geq 0, \tag{CQ}$$

where F^* is the adjoint operator of F and γ is a properly chosen step size satisfying

$$0 < \gamma_k < \frac{2}{\|F\|^2}. \tag{$\gamma 1$}$$

To remove the requirement of knowing $\|F\|$ in ($\gamma 1$), a self-adaptive step size criterion was applied by Yang [8] and enhanced by López et al. [3] as follows

$$\gamma_k = \rho_k \frac{\|(I^{H_1} - P_Q)Fw^k\|^2}{\|F^*(I^{H_1} - P_Q)Fw^k\|^2}. \tag{$\gamma 2$}$$

Recently, several studies [9] - [15] have improved the CQ-algorithm to achieve strong convergence in Hilbert spaces.

In the present article, we develop a new method for approximating the solution of the (SFP) in Hilbert spaces, based on the CQ-algorithm of Byrne [6] and the self-adaptive step size of López et al. [3], and using inertia technique and convex combinations. As a result, our method ensures the strong convergence of the generated sequences, removes the requirement of knowing the norm of transformation operators and enhances convergence speed. A numerical experiment in infinite-dimensional space is given to compare it with the method of Nguyen et al. [9], supporting these claims.

2. Preliminaries

In this section, we introduce some mathematical symbols, definitions, and lemmas which can be used in the proof of our main result.

Let H be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$, and C be a nonempty, closed and convex subset of H. In what follows, we write $x^k \rightharpoonup x$ to indicate that the sequence $\{x^k\}$ converges weakly to x while $x^k \to x$ indicates that the sequence $\{x^k\}$ converges strongly to x. For each $x, y \in H$, it is well known that

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle \leq \|x\|^2 + 2\langle y, x+y\rangle. \tag{1}$$

For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$. This point satisfies $\|x - P_C x\| \leq \|x - u\| \; \forall x, y \in C$. The mapping $P_C : H \to C$ is called the metric projection of H onto C.

Lemma 2.1 (see, [16]). (i) *$P_C x \in C$ for all $x \in H$, and if $x \in C$, then $P_C x = x$.*
(ii) *P_C is a nonexpansive operator of H onto C, i.e., $\|P_C x - P_C y\| \leq \|x - y\|$ $\forall x, y \in C$.*
(iii) *For given $x \in H$ and $y \in C$, $y = P_C x$ if and only if $\langle x - y, z - y\rangle \leq 0$ $\forall z \in C$.*

Let H_0 and H_1 be two real Hilbert spaces, and let $F: H_0 \to H_1$ be a bounded linear operator. An operator $F^*: H_1 \to H_0$, with the property $\langle Fx, y\rangle = \langle x, F^* y\rangle$ for all $x \in H_0$ and $y \in H_1$, is called an adjoint operator of F. The adjoint operator of a bounded linear operator F on a Hilbert space always exists and is uniquely determined. Furthermore, F^* is a bounded linear operator.

Lemma 2.2 (see, [17]). *Assume that $T: C \to H$, with C is a closed and convex subset of a Hilbert space H, is a nonexpansive mapping. Then the mapping $I^H - T$ is demiclosed on C; that is, whenever $\{x^k\}$ is a sequence in C which weakly converges to some point $u^* \in C$, and the sequence $\{(I^H - T)x^k\}$ strongly converges to some y, it follows that $(I^H - T)u^* = y$.*

From Lemma 2.2, if $x^k \rightharpoonup u^*$, and $(I^H - T)x^k \to 0$, then $u^* \in \mathrm{Fix}(T)$.

Lemma 2.3 (see, [18]). *Let $\{s_k\}$ be a real sequence, which does not decrease at infinity in the sense that there exists a subsequence $\{s_{k_n}\}$ such that $s_{k_n} \leq s_{k_n+1}$ $\forall n \geq 0$. Define an integer sequence by $\nu(k) := \max\left\{k_0 \leq n \leq k \mid s_n < s_{n+1}\right\}$, $k \geq k_0$. Then $\nu(k) \to \infty$ as $k \to \infty$ and for all $k \geq k_0$, we have* $\max\{s_{\nu(k)}, s_k\} \leq s_{\nu(k)+1}$.

Lemma 2.4 (see, [7]). *Let $\{s_k\}$ be a sequence of nonnegative numbers satisfying the condition $s_{k+1} \leq (1 - c_k)s_k + c_k b_k$ for all $k \geq 0$, where $\{c_k\}$ is a sequence in $(0,1)$ such that $\sum_{k=1}^{\infty} c_k = \infty$, and $\{b_k\}$ is a sequence of real numbers with $\limsup_{k\to\infty} b_k \leq 0$. Then, $\lim_{k\to\infty} s_k = 0$.*

3. Main Results

In this section, let H_0 and H_1 be real Hilbert spaces, let $C \subset H_0$ and $Q \subset H_1$ be nonempty closed convex subsets. We investigate the (SFP) under the following conditions.

Assumption 3.1.

(A1) $F: H_0 \to H_1$ is a bounded linear operator.

(A2) $f: H_0 \to H_0$ is a contraction mapping with the contraction coefficient $\tau \in [0,1)$.

(A3) The solution set Ω of (SFP) is not empty.

Our algorithm can be expressed as follows.

Algorithm 1

Step 0. Select $\eta \in (0,1)$ and the sequences $\{\alpha_k\}$ such that the condition

$$\{\alpha_k\} \subset (0,1), \ \lim_{k\to\infty} \alpha_k = 0, \text{ and } \sum_{k=1}^{\infty} \alpha_k = \infty, \tag{α}$$

are satisfied. Let $x^0, x^1 \in H_0$ be arbitrary. Set $k := 1$.

Step 1. Compute $w^k = x^k + \theta_k(x^k - x^{k-1})$, where the inertia θ_k is defined by

$$\{\theta_k\} \subset [0,1), \ \lim_{k\to\infty} \frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| = 0. \tag{θ}$$

Step 2. Compute $y^k = w^k - \gamma_k F^*(I^{H_1} - P_Q)Fw^k$, where the step size γ_k is defined by

$$\gamma_k = \rho_k \frac{\|(I^{H_1} - P_Q)Fw^k\|^2}{\|F^*(I^{H_1} - P_Q)Fw^k\|^2 + e_k}, \tag{γ}$$

with the sequences $\{\rho_k\}$ and $\{e_k\}$ are choosen satisfying

$$\{\rho_k\} \subset (0,2), \ \{e_k\} \subset (0,\infty). \tag{ρ}$$

Step 3. Compute $z^k = (1-\eta)y^k + \eta P_C y^k$.
Step 4. Compute $x^{k+1} = \alpha_k f(x^k) + (1-\alpha_k)z^k$.
Step 5. Set $k := k+1$ and go to **Step 1**.

Remark 3.1. It is easy to see that the condition $\lim_{k\to\infty} \frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| = 0$ of (θ) can be implemented easily in the numerical computation as the value of $\|x^k - x^{k-1}\|$ is known before choosing θ_k. Indeed, the parameter θ_k can be chosen such that

$$\theta_k = \begin{cases} \min\left\{\frac{\eta_k}{\|x^k - x^{k-1}\|}, \theta\right\}, & \text{if } x^k \neq x^{k-1}, \\ \theta, & \text{otherwise,} \end{cases} \tag{θ_k}$$

where θ is a constant such that $0 < \theta < 1$ and $\{\eta_k\}$ is a positive sequence such that $\lim_{k\to\infty} \frac{\eta_k}{\alpha_k} = 0$.

Lemma 3.1. *Suppose that all conditions in Assumption 3.1 are satisfied. Let $\{z^k\}$ be a sequence generated by Algorithm 1. Then, for all $u \in \Omega$,*

$$\|z^k - u\|^2 \leq \|w^k - u\|^2 - \gamma_k(2-\rho_k)\left\|(I^{H_1} - P_Q)Fw^k\right\|^2 - \eta(1-\eta)\left\|(I^{H_0} - P_C)y^k\right\|^2.$$

Proof. Since $Fu \in Q$, from Lemma 2.1(iii), we have that

$$\begin{aligned} \|y^k - u\|^2 &= \|w^k - \gamma_k F^*(I^{H_1} - P_Q)Fw^k - u\|^2 \\ &= \|w^k - u\|^2 + \gamma_k^2\|F^*(I^{H_1} - P_Q)Fw^k\|^2 - 2\gamma_k\langle Fw^k - Fu, Fw^k - P_QFw^k\rangle \\ &= \|w^k - u\|^2 + \gamma_k^2\|F^*(I^{H_1} - P_Q)Fw^k\|^2 - 2\gamma_k\|Fw^k - P_QFw^k\|^2 + 2\gamma_k\langle P_QFw^k - Fu, Fw^k - P_QFw^k\rangle \\ &\leq \|w^k - u\|^2 - \gamma_k(2 - \gamma_k\|F\|^2)\|(I^{H_1} - P_Q)Fw^k\|^2 \leq \|w^k - u\|^2 - \gamma_k(2-\rho_k)\|(I^{H_1} - P_Q)Fw^k\|^2. \end{aligned} \tag{2}$$

Since $u \in C$, from the nonexpansive property of P_C, we have:

$$\begin{aligned} &\|y^k - u\|^2 \geq \|P_Cy^k - u\|^2 = \|P_Cy^k - y^k\|^2 + \|y^k - u\|^2 + 2\langle y^k - u, P_Cy^k - y^k\rangle \\ &\Leftrightarrow \langle y^k - u, (I^{H_0} - P_C)y^k\rangle \geq \frac{1}{2}\|(I^{H_0} - P_C)y^k\|^2. \end{aligned} \tag{3}$$

From (2) and (3), we obtain:

$$\begin{aligned} \|z^k - u\|^2 &= \|(1-\eta)y^k + \eta P_Cy^k - u\|^2 \leq \|y^k - u\|^2 - \eta(1-\eta)\|(I^{H_0} - P_C)y^k\|^2 \\ &\leq \|w^k - u\|^2 - \gamma_k(2-\rho_k)\|(I^{H_1} - P_Q)Fw^k\|^2 - \eta(1-\eta)\|(I^{H_0} - P_C)y^k\|^2. \end{aligned}$$

Lemma 3.2. *Suppose that all conditions in Assumption 3.1 are satisfied. Then, the sequence $\{x^k\}$ generated by Algorithm 1 is bounded.*

Proof. Let $u \in \Omega$. Since (θ), there exits $M_1 > 0$ such that

$$\frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| \leq M_1 \quad \forall k \in \mathbb{N}. \tag{M_1}$$

From Lemma 3.1, (M_1) and the contraction property of f with the contraction coefficient $\tau \in [0,1)$, we have:

$$\begin{aligned} &\|z^k - u\| \leq \|w^k - u\| \leq \|x^k - u + \alpha_k\frac{\theta_k}{\alpha_k}(x^k - x^{k-1})\| \leq \|x^k - u\| + \alpha_kM_1 \\ \Rightarrow \|x^{k+1} - u\| &\leq \alpha_k\|f(x^k) - f(u)\| + \alpha_k\|f(u) - u\| + (1-\alpha_k)\|z^k - u\| \\ &\leq \alpha_k\tau\|x^k - u\| + \alpha_k\|f(u) - u\| + (1-\alpha_k)(\|x^k - u\| + \alpha_kM_1) \\ &\leq [1 - \alpha_k(1-\tau)]\|x^k - u\| + \alpha_k(1-\tau)\frac{\|f(u) - u\| + M_1}{1-\tau} \\ &\leq \max\left\{\|x^k - u\|, \frac{\|f(u) - u\| + M_1}{1-\tau}\right\} \leq \cdots \leq \max\left\{\|x^0 - u\|, \frac{\|f(u) - u\| + M_1}{1-\tau}\right\}. \end{aligned}$$

This implies that the sequence $\{x^k\}$ is bounded.

Lemma 3.3. *Suppose that all conditions in Assumption 3.1 are satisfied. Let $\{x^k\}$ be a sequence generated by Algorithm 1. Then, for all $u \in \Omega$,*

$$\|x^{k+1} - u\|^2 \le (1 - c_k)\|x^k - u\|^2 + c_k b_k,$$

where $c_k = \alpha_k(1 - \tau^2)$, and $b_k = \frac{(1-\alpha_k)\theta_k}{(1-\tau^2)\alpha_k}\|x^k - x^{k-1}\|M_2 + \frac{2}{1-\tau^2}\langle f(u) - u, x^{k+1} - u\rangle$, with $M_2 =$ $\sup_{k\ge1}\{2\|x^k - u\| + \alpha_k M_1\}$, *$M_1 > 0$ such that $\frac{\theta_k}{\alpha_k}\|x^k - x^{k-1}\| \le M_1$ for all $k \in \mathbb{N}$.*

Proof. From the definition of M_2, we have:

$$\begin{aligned}\|w^k - u\|^2 &\le \|x^k - u\|^2 + \theta_k^2\|x^k - x^{k-1}\|^2 + 2\theta_k\|x^k - u\|\,\|x^k - x^{k-1}\| \\ &\le \|x^k - u\|^2 + \theta_k\|x^k - x^{k-1}\|M_2. \end{aligned} \tag{4}$$

It follows from Lemma 3.1, (1), (4) and the convexity of $\|\cdot\|^2$ that

$$\begin{aligned}&\|x^{k+1} - u\|^2 = \|\alpha_k\big(f(x^k) - f(u)\big) + (1-\alpha_k)(z^k \quad u) + \alpha_k\,(f(u) - u)\,\|^2 \\ &\le \alpha_k\|f(x^k) - f(u)\|^2 + (1-\alpha_k)\|(z^k - u)\|^2 + 2\alpha_k\langle f(u) - u, x^{k+1} - u\rangle \\ &\le \alpha_k\tau^2\|x^k - u\|^2 + (1-\alpha_k)\Big[\|w^k - u\|^2 - \gamma_k(2-\rho_k)\,\big\|(I^{H_1} - P_Q)Fw^k\big\|^2 - \eta\,(1-\eta)\,\big\|(I^{H_0} - P_C)y^k\big\|^2\Big] \\ &\qquad\qquad + 2\alpha_k\langle f(u) - u, x^{k+1} - u\rangle \\ &= (1-c_k)\|x^k - u\|^2 + c_k\Big[b_k - \frac{1-\alpha_k}{(1-\tau^2)\alpha_k}\Big(\gamma_k(2-\rho_k)\,\big\|(I^{H_1} - P_Q)Fw^k\big\|^2 + \eta\,(1-\eta)\,\big\|(I^{H_0} - P_C)y^k\big\|^2\Big)\Big] \\ &\le (1-c_k)\|x^k - u\|^2 + c_k b_k.\end{aligned}$$

Theorem 3.1. *Suppose that all conditions in Assumption 3.1 are satisfied. Then, the sequence $\{x^k\}$ generated by Algorithm 1 converges strongly to the unique solution u^* of the* VIP($I^{H_0} - f$, Ω)*, which is the $u^* \in \Omega$ satisfying $\big\langle (I^{H_0} - f)u^*, u - u^*\big\rangle \ge 0$ for all $u \in \Omega$.*

Proof. Since f is a contraction mapping, $P_\Omega f$ is a contraction too. By Banach contraction mapping principle, there exists unique point $u^* \in \Omega$ such that $P_\Omega f(u^*) = u^*$. By Lemma 2.1(iii), we obtain u^* is the unique solution to the VIP($I^{H_0} - f$, Ω). Let the u in all preceding lemmas and proofs be replaced by u^*. We will show $\lim_{k\to\infty}\|x^k - u^*\| = 0$ by considering two possible cases.

Case 1. There exists an integer $k_0 \ge 0$ such that $\|x^{k+1} - u^*\| \le \|x^k - u^*\|$ for all $k \ge k_0$. Then, $\lim_{k\to\infty}\|x^k - u^*\|$ exists. From Lemma 3.1, (4) and the convexity of $\|\cdot\|^2$, we get that

$$\begin{aligned}&\|x^{k+1} - u^*\|^2 = \|\alpha_k\big(f(x^k) - u^*\big) + (1-\alpha_k)(z^k - u^*)\|^2 \le \alpha_k\|f(x^k) - u^*\|^2 + (1-\alpha_k)\|z^k - u^*\|^2 \\ &\le \alpha_k\|f(x^k) - u^*\|^2 + (1-\alpha_k)\big(\|x^k - u^*\|^2 + \theta_k\|x^k - x^{k-1}\|M_2\big) \\ &\qquad\qquad (1-\alpha_k)\big[\gamma_k(2-\rho_k)\,\big\|(I^{H_1} - P_Q)Fw^k\big\|^2 + \eta\,(1-\eta)\,\big\|(I^{H_0} - P_C)y^k\big\|^2\big] \\ &\Leftrightarrow (1-\alpha_k)\big[\gamma_k(2-\rho_k)\,\big\|(I^{H_1} - P_Q)Fw^k\big\|^2 + \eta\,(1-\eta)\,\big\|(I^{H_0} - P_C)y^k\big\|^2\big] \\ &\le \alpha_k\|f(x^k) - u^*\|^2 + (1-\alpha_k)\|x^k - u^*\|^2 - \|x^{k+1} - u^*\|^2 + (1-\alpha_k)\alpha_k M_1 M_2.\end{aligned}$$

Since $\lim_{k\to\infty}\|x^k - u^*\|$ exists, $\{f(x^k)\}$ is bounded, $\lim_{k\to\infty}\alpha_k = 0$, $\eta \in (0,1)$, $\rho_k \in (0,2)$ and $\gamma_k \in \Big(0, \frac{2}{\|F\|^2}\Big)$, it follows from the above inequality that

$$\lim_{k\to\infty}\big\|(I^{H_1} - P_Q)Fw^k\big\| = \lim_{k\to\infty}\big\|(I^{H_0} - P_C)y^k\big\| = 0. \tag{5}$$

From (5), the definition of $\{\theta_k\}$ and the boundedness of $\{x^k\}$ and $\{f(x^k)\}$, we have:

$$\begin{cases} \|w^k - x^k\| = \left\|\alpha_k \frac{\theta_k}{\alpha_k}(x^k - x^{k-1})\right\| \to 0 \text{ as } k \to \infty, \\ \|y^k - w^k\| \leq \gamma_k \|F\| \left\|(I^{H_1} - P_Q)Fw^k\right\| \to 0 \text{ as } k \to \infty, \\ \|z^k - y^k\| = \eta \|(I^{H_0} - P_C)y^k\| \to 0 \text{ as } k \to \infty. \end{cases} \tag{6}$$

$$\Rightarrow \|z^k - x^k\| \leq \|z^k - y^k\| + \|y^k - w^k\| + \|w^k - x^k\| \to 0 \text{ as } k \to \infty$$

$$\Rightarrow \|x^{k+1} - x^k\| \leq \alpha_k \|f(x^k) - x^k\| + (1 - \alpha_k)\|z^k - x^k\| \to 0 \text{ as } k \to \infty. \tag{7}$$

Suppose that $\{x^{k_l}\}$ is a subsequence of $\{x^k\}$ such that

$$\limsup_{k\to\infty} \langle f(u^*) - u^*, x^k - u^* \rangle = \lim_{k_l \to \infty} \langle f(u^*) - u^*, x^{k_l} - u^* \rangle. \tag{8}$$

Since $\{x^{k_l}\}$ is bounded, there exits a subsequence $\{x^{k_{l_m}}\}$ of $\{x^{k_l}\}$ which converges weakly to some point $\hat{u}$. Without loss of generality, we may assume that $x^{k_l} \rightharpoonup \hat{u}$. From (6), we get that $\lim_{k_l\to\infty} \|y^{k_l} - x^{k_l}\| = 0$, which implies that $y^{k_l} \rightharpoonup \hat{u}$. Then, from Lemma 2.2 and (5), we obtain $\hat{u} \in \mathrm{Fix}(P_C)$, showing that $\hat{u} \in C$. Moreover, since F is a bounded linear operator and $\lim_{k_l\to\infty} \|y^{k_l} - w^{k_l}\| = 0$, $Fw^{k_l} \rightharpoonup F\hat{u}$. Using Lemma 2.2 and (5) again, we also obtain $F\hat{u} \in Q$, implying that $\hat{u} \in \Omega$. Therefore, since $u^* = P_\Omega f(u^*)$, we deduce from Lemma 2.1(iii), (7) and (8) that

$$\limsup_{k\to\infty} \left\langle f(u^*) - u^*, x^{k+1} - u^* \right\rangle = \limsup_{k\to\infty} \left\langle f(u^*) - u^*, x^k - u^* \right\rangle = \left\langle f(u^*) - P_\Omega f(u^*), \hat{u} - P_\Omega f(u^*) \right\rangle \leq 0.$$

It follows from the above inequality and the definition of $\{b_k\}$ that $\limsup_{k\to\infty} b_k \leq 0$. Moreover, from the definition of $\{c_k\}$, we obtain $\{c_k\} \subset (0,1)$ and $\sum_{k=1}^{\infty} c_k = \infty$. Finally, from Lemma 2.4 and Lemma 3.3, we deduce that $\lim_{k\to\infty} \|x^k - u^*\| = 0$.

Case 2. There exists a subsequence $\{k_l\}$ of $\{k\}$ such that $\|x^{k_l} - u^*\| < \|x^{k_l+1} - u^*\|$ for all $l \geq 0$. Hence, by Lemma 2.3, there exists an integer, nondecreasing sequence $\{v(k)\}$ for $k \geq k_0$ (for some k_0 large enough) such that $v(k) \to \infty$ as $k \to \infty$, $\|x^{v(k)} - u^*\| \leq \|x^{v(k)+1} - u^*\|$ and $\|x^k - u^*\| \leq \|x^{v(k)+1} - u^*\|$ for each $k \geq 0$. From Lemma 3.3 with k replaced by $v(k)$, we have:

$$0 \leq \|x^{v(k)+1} - u^*\|^2 - \|x^{v(k)} - u^*\|^2 \leq -c_{v(k)} + c_{v(k)} b_{v(k)}.$$

Since $\{x^{v(k)}\}$ is bounded, $\lim_{k\to\infty} \alpha_{v(k)} = 0$ and $\lim_{k\to\infty} \frac{\theta_{v(k)}}{\alpha_{v(k)}} \|x^{v(k)} - x^{v(k)-1}\| = 0$, it follows from the above inequality that:

$$\lim_{k\to\infty} \left(\|x^{v(k)+1} - u^*\|^2 - \|x^{v(k)} - u^*\|^2 \right) = 0. \tag{9}$$

Using the above equality, by similar argument to Case 1, we obtain: $\lim_{k\to\infty} \left\|(I^{H_1} - P_Q)Fw^{v(k)}\right\| = \lim_{k\to\infty} \left\|(I^{H_0} - P_C)y^{v(k)}\right\| = 0$. Also we get

$$\begin{aligned} \|x^{v(k)} - u^*\| \leq \|x^{v(k)+1} - u^*\|^2 &\leq \left(1 - \alpha_{v(k)}(1 - \tau^2)\right) \|x^{v(k)} - u^*\|^2 \\ &\quad + (1 - \alpha_{v(k)})\theta_{v(k)} \|x^{v(k)} - x^{v(k)-1}\| M_2 + 2\alpha_{v(k)} \langle f(u^*) - u^*, x^{v(k)+1} - u^* \rangle \\ \Rightarrow (1 - \tau^2)\|x^{v(k)} - u^*\|^2 &\leq (1 - \alpha_{v(k)}) \frac{\theta_{v(k)}}{\alpha_{v(k)}} \|x^{v(k)} - x^{v(k)-1}\| M_2 + 2\langle f(u^*) - u^*, x^{v(k)+1} - u^* \rangle, \end{aligned}$$

where $\limsup_{k\to\infty} \langle f(u^*) - u^*, x^{v(k)+1} - u^* \rangle \leq 0$. Therefore, we have that $\lim_{k\to\infty} \|x^{v(k)} - u^*\|^2 = 0$, which together with (9) that $\lim_{k\to\infty} \|x^{v(k)+1} - u^*\|^2 = 0$. Hence, since $\|x^k - u^*\| \leq \|x^{v(k)+1} - u^*\|$, we get $\lim_{k\to\infty} \|x^k - u^*\|^2 = 0$. This complete the proof.

Example 3.1. A numerical experiment on infinite-dimensional spaces is given to illustrate the effectiveness of our method and compare it with [9]. The computations were performed using Python on a Dell Latitude E6540 laptop with an Intel Core i7-4800MQ @ 2.70 GHz processor and 16 GB of RAM.

Let $H_0 = H_1 = (l^2, \|\cdot\|_{l^2})$, where $l^2 := \{x = (x_1, x_2, x_3, \ldots), x_i \in \mathbb{R}\colon \sum_{i=1}^{\infty} x_i^2 < \infty\}$ and $\|x\|_{l^2} := (\sum_{i=1}^{\infty} x_i^2)^{\frac{1}{2}}$ $\forall x \in l^2$. Let $C = \{x \in l^2\colon \|x - (1,0,0,0,\ldots)\| \leq 1\} \in H_0$ and $Q = \{y \in l^2\colon \|y - (-1,0,0,0,\ldots)\| \leq 3\} \in H_1$. The bounded linear operator $F\colon H_0 \to H_1$ and the contraction operator $f\colon H_0 \to H_0$ are determined by $Fx = \frac{2}{3}x$ and $f(x) = \frac{1}{20}x$ for all $x \in l^2$. Then, we have a VIP$(I^{H_0} - f, \Omega)$. In theory, both Algorithm 1 and the algorithm of Nguyen et al. [9] strongly converge to the solution of this problem, which is the sequence $(0,0,0,\ldots)$.

For convenience, we denote Algorithm 1 by Current Alg., and the algorithm of Nguyen et al. [9] by Alg. of Nguyen. The stopping criteria is that $\|x^k - u^*\| \leq \varepsilon$, where ε is a chosen tolerance. The initial values and parameters of these methods are chosen as follows

- Case 1: $x^0 = \frac{1}{2}x^1 = (\frac{50}{1^2}, \frac{50}{2^2}, \frac{50}{3^2}, \ldots)$,
  - The Alg. of Nguyen: $\beta = 0.54, \alpha_k = \frac{1}{k+51}, \rho_k = 0.1, \kappa_k = 5,$ where $k = 1,2,3,\ldots.$
  - Current Alg.: $\eta = 0.54, \alpha_k = \frac{1}{k+51}, \rho_k = 0.1, e_k = 5, \theta_k$ satisfies (θ_k) with $\theta = 0.44$ and $\eta_k = \frac{1}{(k+51)^{1.1}}$, where $k = 1,2,3,\ldots.$
- Case 2: $x^0 = 3x^1 = (\frac{120}{1^{1.5}}, \frac{120}{2^{1.5}}, \frac{120}{3^{1.5}}, \ldots)$,

Table 1. *Numerical results for two algorithms in two cases of initial values and parameters*

		Case 1		Case 2	
ε		Alg. of Nguyen	Current Alg.	Alg. of Nguyen	Current Alg.
10^{-1}	Time (s)	0.788546	0.419758	0.012992	0.003998
	Iter. (k)	1385	324	26	5
10^{-2}	Time (s)	8.218267	1.328235	0.197886	0.022983
	Iter. (k)	16151	1399	287	29
10^{-3}	Time (s)	93.946898	5.468849	1.328234	0.064960
	Iter. (k)	182841	5578	3336	75

  - The Alg. of Nguyen: $\beta = 0.71, \alpha_k = \frac{1}{k}, \rho_k = 0.5, \kappa_k = 2,$ where $k = 1,2,3,\ldots.$
  - Current Alg.: $\eta = 0.71, \alpha_k = \frac{1}{k}, \rho_k = 0.5, e_k = 2, \theta_k$ satisfies (θ_k) with $\theta = 0.95$ and $\eta_k = \frac{1}{k^{1.01}}$, where $k = 1,2,3,\ldots.$

Since both algorithms use similar initial values and parameters, adjusting the inertia enables our algorithm to achieve greater efficiency than the algorithm of Nguyen and Tran [9]. The results are shown in Table 1, demonstrating that our algorithm significantly outperforms theirs in both execution time and the number of iterations, especially as the tolerance decreases.

4. Conclusion

In this paper, we have described an iterative method (Algorithm 3) for approximating the split feasibility problem (SFP) in real Hilbert spaces, based on the *CQ*-algorithm of Byrne and the self-adaptive step size of López et al., and using inertia technique and convex combinations. By effectively incorporating convex combinations into the algorithm, we have proved the strong convergence results of the suggested method with variable step size under mild conditions on the control parameters (Theorem 3.1). The application of the inertial technique has improved the convergence speed of our method, outperforming a related one in both execution time and the number of iterations, especially as the tolerance decreases, as demonstrated by a numerical experiment in infinite-dimensional space (Example 3.1). Furthermore, the ability to solve a class of variational inequality problems over the solution set of the split feasibility problem has expanded the practical applicability of our method.

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