DETERMINE THE SOLUTIONS IN A SINGLE-DEGREE-OF-FREEDOM NONLINEAR SYSTEM AND APPLYING THE WEIGHTED AVERAGE METHOD

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ARTICLE INFO

Received: **23/10/2024**
Revised: **26/11/2024**
Published: **27/11/2024**

KEYWORDS

Nonlinear
Weighted average
Numerical solution
Oscillation frequency
Least squares

ABSTRACT

This paper proposes the determination of the solution of a single-degree-of-freedom nonlinear system. The contents of the paper are to calculate the natural frequencies of the nonlinear system using three methods. They are the method of direct integration of the differential equation, the method based directly on the numerical calculation results and the method of using the weighted average function. This paper used the least square criterion to evaluate the error between the numerical methods and the proposed methods. The results obtained are the exact natural frequencies and the frequencies according to the proposed methods. Along with the frequency calculation results, the paper also obtained the solutions as approximate analytical expressions. From obtained results, it can be seen that the errors between the solutions according to the numerical method and the solution according to the proposed methods are very small. The weighted average method to determine the solution of the nonlinear differential equation revealed that this is an approach with many advantages. The weighted average method can be easy to apply to the calculation and obtain results with high reliability.

XÁC ĐỊNH NGHIỆM CỦA HỆ PHI TUYẾN MỘT BẬC TỰ DO VÀ ỨNG DỤNG PHƯƠNG PHÁP TRUNG BÌNH CÓ TRỌNG SỐ

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THÔNG TIN BÀI BÁO

Ngày nhận bài: **23/10/2024**
Ngày hoàn thiện: **26/11/2024**
Ngày đăng: **27/11/2024**

TỪ KHÓA

Phi tuyến
Trung bình trọng số
Phương pháp số
Tần số dao động
Bình phương tối thiểu

TÓM TẮT

Bài báo này đề xuất việc xác định nghiệm của hệ phi tuyến một bậc tự do. Nội dung bài báo là tính toán tần số dao động riêng của hệ phi tuyến bằng ba phương pháp: Đó là, phương pháp tích phân trực tiếp phương trình vi phân, phương pháp dựa trực tiếp vào kết quả tính toán số và phương pháp sử dụng hàm trung bình có trọng số. Bài báo đã sử dụng tiêu chuẩn bình phương tối thiểu để đánh giá sai số giữa phương pháp số và phương pháp đề xuất. Kết quả nhận được là tần số chính xác và tần số theo phương pháp đề xuất. Cùng với kết quả tính toán tần số, bài báo cũng nhận được nghiệm là biểu thức giải tích gần đúng. Từ kết quả đó, nhận thấy sai số giữa nghiệm theo phương pháp số và nghiệm theo phương pháp đề xuất là rất nhỏ. Ứng dụng phương pháp trung bình có trọng số để xác định nghiệm của phương trình vi phân phi tuyến cho thấy đây là cách tiếp cận có nhiều ưu điểm vì sự dễ dàng vận dụng vào tính toán, đồng thời kết quả nhận được có độ tin cậy cao.

DOI: https://doi.org/10.34238/tnu-jst.11385

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1. Introduction

Methods for finding solutions to nonlinear equations include the perturbation methods [1], the asymptotic methods [2], the homotopy analysis method [3], the homotopy perturbation method [4], the variational iteration method [5], the Gamma function method [6], the weighted average function methods [7] - [9].

The results of approximate analytical methods often have solutions of the first order or higher approximation. The results of solutions in the first order approximation are often chosen in the form of Acos(ωt), where A is the amplitude and ω is the oscillation frequency. Finding the oscillation frequency ω using analytical methods [1] - [3] is sometimes complicated to perform and the results may have large errors, leading to the results of the first order approximation that may not ensure the necessary accuracy. In the approximate analytical methods [1] - [5] and [6] the results in the first order iteration often have large errors compared to the exact solution. So, the need to find methods that results in the first iteration are reliable is meaningful and worth considering.

This paper proposes the methods of determining the solutions of a nonlinear system through the results of numerical calculations and the method of using the weighted average function (WAM) [7] - [9] to obtain the results of the approximate analytical solution. The obtained results are promising and reliable because the errors are very small, and the calculation process is relatively simple, easy to apply to solving some nonlinear equations when applying the calculation in the first order iteration.

The contents of this paper include: 1. Introduction; 2. Methods; 3. Results and discussions; 4. Conclusion.

2. Methods

It is supposed that the following nonlinear equation is considered to solve:

$$(1+\varepsilon_1 u^2)\ddot{u}(t)+\varepsilon_2 u(t)+\varepsilon_3 (u(t))^3=0 \tag{1}$$

with initial conditions

$$u(0)=A, \dot{u}(0)=0 \tag{2}$$

where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are positive or non-negative real numbers. In the special case when $\varepsilon_1=0$, equation (1) becomes briefly:

$$\ddot{u}+\varepsilon_2 u+\varepsilon_3 u^3=0 \tag{3}$$

With the special cases when $\varepsilon_1=1, \varepsilon_2=\varepsilon_3=\varepsilon$, equation (1) becomes:

$$\ddot{u}+\varepsilon u=0 \tag{4}$$

This paper will solve the cases of equations by the formulas (1) and (3). Solving nonlinear differential equations (1), (3) by other analytical methods [1] - [3] is relatively complicated and difficult to apply. In this paper, the following methods are implemented: First, the direct integration method is applied to find the exact natural frequency ω_{ex} and then solved by the numerical method to get numerical results. From there, we establish a criterion for evaluating the errors between the solutions by the numerical method and the proposed solutions by the expression as Acos(ωt). Afterward, based on the method of using the weighted average function we will determine the frequency values and the approximate analytical solutions of the above nonlinear equations. The results are then compared to evaluate the errors to clarify the reliability of the proposed methods. The specific contents are implemented in the sections below.

2.1. Determine the exact frequency ω_{ex} by the direct integration method

Equation (1) is rewritten in the form

$$\ddot{u}+\frac{\varepsilon_2 u+\varepsilon_3 u^3}{1+\varepsilon_1 u^2}=0 \tag{5}$$

Setting:

$$v=\dot{u}=\frac{du}{dt}, \dot{v}=\frac{dv}{dt}=\frac{dv}{du}\frac{du}{dt}=v\frac{dv}{du} \tag{6}$$

Substitute equation (6) into equation (5) yields

$$v\frac{dv}{du}+\frac{\varepsilon_2 u+\varepsilon_3 u^3}{1+\varepsilon_1 u^2}=0, \to vdv=-\frac{\varepsilon_2 u+\varepsilon_3 u^3}{1+\varepsilon_1 u^2}du \tag{7}$$

Integrating both sides of equation (7) gives

$$\frac{v^2}{2}=H-F(u) \tag{8}$$

where H is the integration constant and F(u) is determined by the following expression:

$$F(u)=\int\frac{\varepsilon_2 u+\varepsilon_3 u^3}{1+\varepsilon_1 u^2}du=\frac{1}{2\varepsilon_1}\left[\varepsilon_3 u^2+\left(\varepsilon_2-\frac{\varepsilon_3}{\varepsilon_1}\right)ln(1+\varepsilon_1 u^2)\right] \tag{9}$$

Substitute the formula (9) into the expression (8) to get

$$\frac{v^2}{2}=H-\frac{1}{2\varepsilon_1}\left[\varepsilon_3 u^2+\left(\varepsilon_2-\frac{\varepsilon_3}{\varepsilon_1}\right)ln(1+\varepsilon_1 u^2)\right] \tag{10}$$

The constant of H can be gotten by the initial condition (2). When $t=0, u(0)=A, v=\dot{u}(0)=0$ so from the formula (10) we get the value

$$H=\frac{1}{2\varepsilon_1}\left[\varepsilon_3 A^2+\left(\varepsilon_2-\frac{\varepsilon_3}{\varepsilon_1}\right)ln(1+\varepsilon_1 A^2)\right] \tag{11}$$

Substitute the expression (11) into the expression (10) to get

$$v^2\equiv\left(\frac{du}{dt}\right)^2=\frac{1}{\varepsilon_1}\left[\varepsilon_3(A^2-u^2)+\left(\varepsilon_2-\frac{\varepsilon_3}{\varepsilon_1}\right)ln(\frac{1+\varepsilon_1 A^2}{1+\varepsilon_1 u^2})\right] \tag{12}$$

The formula (12) can get as

$$dt=\frac{\pm\sqrt{\varepsilon_1}du}{\sqrt{\varepsilon_3(A^2-u^2)+\left(\varepsilon_2-\frac{\varepsilon_3}{\varepsilon_1}\right)ln(\frac{1+\varepsilon_1 A^2}{1+\varepsilon_1 u^2})}} \tag{13}$$

We could see that the closed trajectory is symmetric with respect to the u axis [1]. So, the time needed for the representative point to move from $u=-A$ to $u=A$ is one-half the period T, then (13) will be integrated and becomes

$$T=2\sqrt{\varepsilon_1}\int_{-A}^{A}\frac{du}{\sqrt{\varepsilon_3(A^2-u^2)+\left(\varepsilon_2-\frac{\varepsilon_3}{\varepsilon_1}\right)ln\left(\frac{1+\varepsilon_1 A^2}{1+\varepsilon_1 u^2}\right)}} \tag{14}$$

Additionally, in (14), the integral over the range $[-A,0]$ is the same as that over the interval $[0,A]$, then gets

$$T=T_{ex}=4\sqrt{\varepsilon_1}\int_{0}^{A}\frac{du}{\sqrt{\varepsilon_3(A^2-u^2)+\left(\varepsilon_2-\frac{\varepsilon_3}{\varepsilon_1}\right)ln\left(\frac{1+\varepsilon_1 A^2}{1+\varepsilon_1 u^2}\right)}}, \omega_{ex}=\frac{2\pi}{T_{ex}}, \tag{15}$$

where T_{ex} is the period and ω_{ex} is the oscillation frequency. The expression (15) is the exact one of the period and oscillation frequency of the original equation (1).

The exact solution of the original nonlinear equation (3) will be gotten as follows

$$\omega_{ex(3)}=\frac{2\pi}{T_{ex(3)}}, \quad T_{ex(3)}=4\sqrt{2}\int_0^A\frac{du}{\sqrt{\varepsilon_3\left(A^4-u^4\right)+\varepsilon_2\left(A^2-u^2\right)}} \tag{16}$$

After some manipulations the expression (16) is written in the form [1], [4]:

$$\omega_{\mathrm{ex}(3)} = \frac{2\pi}{T_{(3)}}, \quad T_{\mathrm{ex}(3)} = \frac{4}{\sqrt{\varepsilon_2 + \varepsilon_3 A^2}} \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k\sin^2\theta}}, \quad k = \frac{\varepsilon_3 A^2}{2(\varepsilon_2 + \varepsilon_3 A^2)} \tag{17}$$

2.2. The approximate solution based on results of the numerical method

By analyzing with the odd-order nonlinear systems as same as in [10], the oscillation system should be described under the forms of $cos[(2k+1)\omega t)]$. Because of that, the original nonlinear equations (1) and (3) could be proposed by the solution as

$$u_{pro1}(t) = A\,cos(\,\omega_{pro1}t) \tag{18}$$

where ω_{pro1} is the unknown frequency that needs to be found.

We will solve the original nonlinear equation (1) or (3) by the numerical method, for example using programs as Matlab, Mathematica or Maple. The softwares often use the 4th-5th order Runge-Kutta algorithm. The result of solving the equation (1) or (3) by the numerical method is denoted as

$$u(t) = u_{num}(t) \tag{19}$$

To determine the unknown frequency ω_{pro1}, the error between the solutions (18) and (19) has been established by the form

$$err = A\,cos(\,\omega_{pro1}t) - u_{num}(t) \tag{20}$$

Let the error (20) reaches its minimum according to the following least squares criterion

$$Min(\,(err)^2) = \frac{1}{N+1}\sum_{i=0}^{N}\left(A\,cos(\,\omega_{pro1}t_i) - u_{num}(t_i)\right)^2, t_i = \frac{t_1 - t_2}{N} \tag{21}$$

In the formula (21), the time range is considered to be $t = [t_1 \div t_2]$; then, $t_i = \frac{t_2 - t_1}{N}$ are the time-divided pieces, and N is the total number of pieces. The larger N and the smaller t_i are, the more accurate the results are. From the criterion (21), the unknown frequency ω_{pro1} has been determined by the numerical method.

2.3. The approximate solution by the weighted average method (WAM)

Similar to the expression (18) and then applying the weighted average method (WAM) [7], we will solve the system (1) when the approximate solution of (1) is written in the form

$$u_{pro2}(t) = (A - B)\cos(\omega_{pro2}t) + Bcos(3\omega_{pro2}t) \tag{22}$$

where B is the amplitude and ω_{pro2} is the frequency. The amplitude B is often a given value and the frequency ω_{pro2} is an unknown value which need to find. Note that the solution expression (22) always satisfies the initial conditions (2).

The average value function W of a function x(τ) with a time period τ=ωt is determined by the following formula [7] as

$$W(x(\tau)) = \int_0^\infty s^2\omega^2 te^{-s\omega t}x(\omega t)\,dt = \int_0^\infty s^2\tau e^{-s\tau}x(\tau)\,d\tau \tag{23}$$

In formula (23) the parameter s is a positive number. According to the Galerkin method [9], we will set the following average criterion as

$$\langle[(1 + \varepsilon_1 u^2)\ddot{u}_{pro2}(t) + \varepsilon_2 u_{pro2}(t) + \varepsilon_3 u_{pro2}^3(t)].u_{pro2}(t)\rangle = 0 \tag{24}$$

where the symbol $\langle . \rangle$ is the average value, taken according to the expression (23).

By substituting (21) into (24) and then applying the average value formula (23), the formula of the frequency ω will be determined. In the case when A = 10, B = 0.04746512525, ϵ_1 = 0.1, ϵ_2 = 1, ϵ_3 = 0.1, the formula of the frequency $\omega = \omega_{pro2}$ can be obtained as follows:

$$\omega^2 \equiv \omega_{pro2}^2 = \frac{TS}{MS} \tag{25}$$

where

$$TS = (1045.252573wcos(2) + 4740.811914 + 259.8311004wcos(4) + \tag{26}$$

$$4.747319708wcos(6) + 0.03358045352wcos(8) + 0.0001064284168wcos(10) + 1.268932538\,10^{-7}wcos(12))$$

and

$$MS = (-563.6879051wcos(2) - 2547.921689 - 146.1469827wcos(4) - 7.196377659wcos(6) - 0.08405756234wcos(8) - 0.0003724994589wcos(10) - 5.710196421\,10^{-7}wcos(12)) \quad (27)$$

where $wcos(i), (i = 2,4,\dots,12)$ is the weighted average value calculated by the formula (23) with $x(\tau) = cos(i\omega t)$. According to the frequency formula (25), $\omega = \omega_{pro2}$ depends on the parameter value s, that is $\omega_{pro2} = \omega(s)$. To determine the optimal parameter s, let the square of the residual errors reach the smallest value, these residuals are the values when substituting the solution expression (22) into the original nonlinear equation (1) or (3)

$$Err^2 = \frac{1}{N+1}\sum_{i=0}^{N}\left[(1+\varepsilon_1 u^2)\ddot{u}_{pro2}(t_i) + \varepsilon_2 u_{pro2}(t_i) + \varepsilon_3 u_{pro2}^3(t_i)\right]^2 \to min \quad (28)$$

where t_i, N are described in the formula (21).

3. Results and discussions

3.1. Calculate oscillation frequencies

Tables 1 to 6 show the results of exact frequency calculations based on section 2.1 and numerical calculations based on section 2.2. The results in tables 1 to 6 are different cases corresponding to the changes of parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and A.

Table 1. *Values of frequency in equation (1) when* $\varepsilon_1 = 0.1, \varepsilon_2 = 1, \varepsilon_3 = 1$

A	1	5	10	30	100	200
ω_{ex}	1.2704	2.56827078	2.95396608	3.13372712	3.15951876	3.16157720
ω_{pro1}	1.2715	2.567863202	2.953800876	3.133718728	3.159519457	3.161577398
Error(%)	0.0793	0.01587	0.00559	0.00027	0.00002	0.00001

Table 2. *Values of frequency in equation (1) when* $\varepsilon_1 = 0.5, \varepsilon_2 = 2, \varepsilon_3 = 5$

A	1	5	10	30	100	200
ω_{ex}	2.02417697	3.009705	3.11822951	3.15691053	3.16177875	3.16215205
ω_{pro1}	2.0236447	3.009577	3.11820647	3.15691159	3.161778879	3.162152064
Error(%)	0.0262979	0.0042365	0.0007390	0.0000334	0.0000040	0.0000002

Table 3. *Values of frequency in equation (1) when* $\varepsilon_1 = 5, \varepsilon_2 = 3, \varepsilon_3 = 0.5$

A	1	5	10	30	100	200
ω_{ex}	0.86933	0.3738363	0.33282756	0.31821973	0.31641021	0.316273507
ω_{pro1}	0.87024	0.3755275	0.332832654	0.318173041	0.316405220	0.3162723046
Error(%)	0.1048	0.4503	0.0015	0.0147	0.0016	0.0004

Table 4. *Values of frequency in equation (3) when* $\varepsilon_2 = 1, \varepsilon_3 = 1$

A	1	5	10	50	100	200
$\omega_{ex(3)}$	1.31777606	4.35746185	8.53358619	42.3729955	84.7274799	169.445702
$\omega_{pro1(3)}$	1.31829683	4.35743356	8.53328455	42.3730847	84.7280341	169.4477032
Error(%)	0.03950	0.00065	0.00353	0.00021	0.00065	0.00118

Table 5. *Values of frequency in equation (3) when* $\varepsilon_2 = 2, \varepsilon_3 = 5$

A	1	5	10	50	100	200
$\omega_{ex(3)}$	2.37629671	9.58178803	18.9993678	94.7323420	189.448125	378.887970
$\omega_{pro1(3)}$	2.37604332	9.58157554	18.9992552	94.7330285	189.450632	378.8980666
Error(%)	0.0107	0.0022	0.0006	0.0007	0.0013	0.0027

Table 6. *Values of frequency in equation (3) when* $\varepsilon_2 = 3, \varepsilon_3 = 0.5$

A	1	5	10	50	100	200
$\omega_{ex(3)}$	1.836644	3.474749	6.24617417	30.0058209	59.9331894	119.827114
$\omega_{pro1(3)}$	1.8366900	3.474197	6.246806561	30.00579543	59.93339635	119.8281100
Error(%)	0.0025	0.0159	0.0101	0.0001	0.0003	0.0008

From the results in the Table 1 to Table 6, it can be seen that:

- The exact frequency values are all expressed by numerical results according to formulas (15), (16), (17). The frequency values according to the proposed method ω_{pro1} have very small errors compared to the exact frequency values.
- The approximate solution in the form of the formula $Acos(\omega_{pro1}t)$ only needs to be performed in one iteration, so it is easy to apply to nonlinear systems.
- The amplitude values A in Table 1, 2 and 3 are the solutions in equation (1) when values of A are very large. The frequency values do not change much in comparing between them with each other.

3.2. Calculated by the numerical method and the WAM

Based on the sections 2.2 and 2.3 above, we will get the solutions u(t) and frequencies according to the formulas (18), (22) and (25). The calculation results for some cases are shown in Tables 7 and 8. The results in Table 7 show that the frequency errors between values calculated by the two proposed methods and by the frequency exact are only under 0.0892%. The displacement u(t) errors shown in Table 8 between values calculated by the two proposed methods and by the numerical method are also under 0.00229. It is seen that the results by the proposed methods are very reliable.

Table 7. *Values of frequency in equation (1) obtained in the sections 2.2 and 2.3*

A	ω_{exact}	ω_{pro1}	Error (%)	ω_{pro2}	Error (%)
A=10	1.363577568	1.363524025	0.00392	1.36236097	0.0892
A=20	1.399161152	1.399111748	0.0035	1.398274159	0.063
A=30	1.407147271	1.407120147	0.0019	1.406732425	0.0294
A=40	1.410130644	1.410113921	0.0012	1.410166035	0.0025

Table 8. *Values of u(t) in equation (1) obtained in the sections 2.2 and 2.3*

Cases of changing	$u_{pro1}(t)$	Error $u_{pro1}(t)$ with $u_{num}(t)$	$u_{pro2}(t)$	Error $u_{pro2}(t)$ with $u_{num}(t)$
A=10,$\epsilon_1 = 0.1$ $\epsilon_2 = 1, \epsilon_3 = 0.2$	10cos(1.363524025t)	0.00229	9.95cos(1.3623t) + 0.05cos(4.087t)	0.00059
A=20,$\epsilon_1 = 0.1$ $\epsilon_2 = 1, \epsilon_3 = 0.2$	20cos(1.399111748t)	0.001348	19.96cos(1.398t) + 0.04cos(4.194t)	0.00115
A=30, $\epsilon_1 = 0.1$ $\epsilon_2 = 1, \epsilon_3 = 0.2$	30cos(1.407120147t)	0.00080	29.97cos(1.4067t) + 0.03cos(4.220t)	0.000577
A=40, $\epsilon_1 = 0.1$ $\epsilon_2 = 1, \epsilon_3 = 0.2$	40cos(1.410113921t)	0.0005	39.98cos(1.410t) + 0.02cos(4.230t)	0.00005

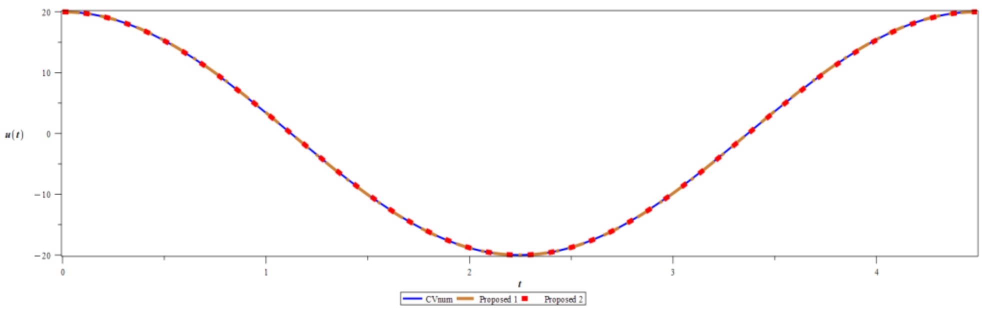

Figure 1. *Solution of u(t) when* $A = 20, \epsilon_1 = 0.1, \epsilon_2 = 1, \epsilon_3 = 0.2$

Figure 1 and Fgiure 2 show the solutions of displacement u(t) calculated by formulas (18) and (22) – calculated by the two proposed methods and compared with the displacement calculated by the numerical method. These figures show that they are very closed.

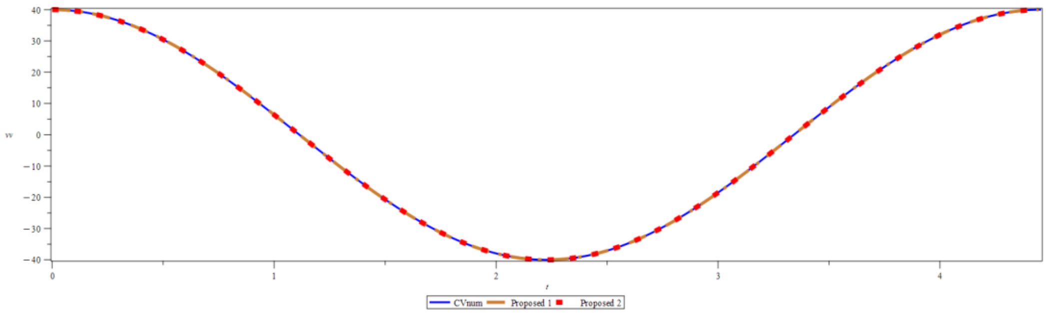

Figure 2. *Solution of u(t) when* $A = 40, \epsilon_1 = 0.1, \epsilon_2 = 1, \epsilon_3 = 0.2$

4. Conclusion

This paper has solved the single-degree-of-freedom nonlinear system to obtain the natural frequencies and solutions of displacement u(t). The paper has contributed to the analysis of the system in the form of equation (1), which is a new system that has not been considered in previous documents. However, the special cases in the form of equation (3), the calculation results here are similar to the previous results. The calculation results have determined the exact frequency by the direct integration method. Based on the numerical method, the paper has determined the frequency when using the error criterion to reach the minimum. The method using the weighted average function obtains both the frequency and the approximate analytical solution containing the expressions of $\cos(\omega t)$ and $\cos(3\omega t)$. The obtained results in the paper on the oscillation frequency have very reliable values because the error with the exact solution is very small. The methods of determining the solutions here can be applied in practice when the nonlinear systems are complex.

REFERENCES

[1] A. H. Nayfeh, *Introduction to Perturbation Techniques*. JohnWiley & Sons, New York, 1981.

[2] Y. A. Mitropolsky and V. D. Nguyen, *Applied asymptotic methods in nonlinear oscillations*. Springer Science & Business Media, 2013.

[3] T. H. Duong and T. T. Nguyen, "Research on nonlinear free vibrations using homotopy analysis method," *TNU Journal of Science and Technology*, vol. 229, no. 06, pp. 321-329, 2024.

[4] J. H. He, "Homotopy perturbation technique," *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3-4, pp. 257-262, 1999.

[5] J. H. He, "Variational iteration method–a kind of non-linear analytical technique: some examples," *International Journal of Non-Linear Mechanics*, vol. 34, no. 4, pp. 699-708, 1999.

[6] K. J. Wang and G. D. Wang, "Gamma function method for the nonlinear cubic-quintic Duffing oscillators," *Journal of Low Frequency Noise, Vibration Active Control*, vol. 41, no. 1, pp. 216-222, 2022.

[7] D. A. Nguyen, "Dual approach to averaged values of functions: A form for weighting coefficient," *Vietnam Journal of Mechanics*, vol. 37, no. 2, pp. 145-150, 2015.

[8] D. A. Nguyen, Q. H. Ninh, and V. H. Dang, "The equivalent linearization method with a weighted averaging for analyzing of nonlinear vibrating systems," *Latin American Journal of Solids Structures*, vol. 14, no. 9, pp. 1723-1740, 2017.

[9] T. A. Nguyen and D. A. Nguyen, "A modified averaging operator with some applications," *Vietnam Journal of Mechanics*, vol. 42, no. 3, pp. 341-354, 2020.

[10] R. Mickens and D. Semwogerere, "Fourier analysis of a rational harmonic balance approximation for periodic solutions," *Journal of Sound and Vibration*, vol. 195, no. 3, pp. 528-550, 1996.