ON THE ENERGY EQUALITY OF THE NAVIER – STOKES EQUATIONS IN BOUNDED THREE DIMENSIONAL DOMAINS

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Abstract

The energy equality

$$\frac{1}{2}\|u(t)\|_2^2+\int_0^t\|\nabla u\|_2^2 d\tau=\frac{1}{2}\|u_0\|_2^2$$

is an open problem for the Navier-Stokes equations. In this paper we present a condition for the energy equality of weak solutions to the Navier – Stokes equations in bounded three dimensional domains. We prove that the energy equality holds for weak solutions in the functional class $L^3(0,T;V^s)$ $\left(s\geq\frac{5}{6}\right)$.

Keywords: Navier – Stokes equations, weak solutions, energy equality, energy inequality, bounded domain.

1. Introduction

We consider the three dimentional initial boundary value problem for the Navier – Stokes equations

$$\frac{\partial u}{\partial t}-\Delta u+(u\cdot\nabla)u+\nabla p=0$$

in $\Omega^T:=(0,T)\times\Omega, i=\overline{1,3}$

(1)

$$div(u)=\sum_{i=1}^{3}\frac{\partial u_i}{\partial x_i}=0\ \ in\ \ \Omega^T$$

$$u(0,x)=u_0(x)\ \ in\ \ \Omega$$

where Ω is a smooth bounded domain in $\mathbb{R}^3, u_0(x)$ are given functions with $u_0(x)$ satisfying the condition $div(u_0)=0$.

We recall the definition of weak solutions.

Definition 1.1. *A vector field*

$$u\in L^\infty\left(0,T;L_\sigma^2(\Omega)\right)\cap L_{loc}^2(0,T;W_0^{1,2}(\Omega))$$

is called a weak solution of the Navier – Stokes equations if the relation

$$-(u,w_t)_{\Omega,T}+(\nabla u,\nabla w)_{\Omega,T}-(uu,\nabla w)_{\Omega,T}=\left(u_0,w(0)\right)\_\Omega$$

is satisfied for all test functions $w\in C_0^\infty\left(0,T;C_{0,0}^\infty(\Omega)\right)$.

In this definition $(.,.)_\Omega$ means the usual pairing of functions on Ω, $(.,.)_{\Omega,T}$ means the corresponding pairing on $[0,T)\times\Omega$. Finally $uu=\left(u_iu_j\right)_{i,j=1}^3$ for $u=(u_1,u_2,u_3)$ such that $u.\nabla u=(u.\nabla)u=div(uu)$ when $div(u)=0$.

Leray[3] and Hopf[2] showed the global existence of weak solutions to Navier – Stokes equations satisfying the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2+\int_0^t\|\nabla u\|_2^2 d\tau\leq\frac{1}{2}\|u_0\|_2^2$$

for all $t\in[0,T)$.

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However, the energy equality of weak solutions

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2 d\tau = \frac{1}{2}\|u_0\|_2^2$$

is still an open problem. Serrin[4] showed that if a weak solution u belongs to $L^s(0,T;L^q(\Omega))$ for some $q > 3, s > 4$ with

$$\frac{3}{q} + \frac{2}{s} \leq 1$$

then energy equality holds. Later, Shinbrot[5] derived the same conclusion if the weak solution u belongs to $L^s(0,T;L^q(\Omega))$ for some $s > 2, q \geq 4$ with

$$\frac{2}{q} + \frac{2}{s} \leq 1.$$

Sohr[6] proved the energy equality for weak solutions if uu belongs to $L^2_{loc}(0,T;L^2(\Omega)^{n^2}$.

In the present paper we prove that the energy equality holds for weak solutions in the functional class $L^3(0,T;V^s)$ $\left(s \geq \frac{5}{6}\right)$. We have $L^3(0,T;V^s) \subset L^3(0,T;L^p)$ $\left(p \geq \frac{9}{2}\right)$ with

$$\frac{3}{p} + \frac{2}{3} \leq \frac{4}{3}.$$

2. Preliminaries

In this section we briefly recall some standard facts. Let $\mathbb{P}: L^2(\Omega) \to L^2_\sigma(\Omega)$ be the L^2- orthogonal projection. Let A be the Stokes operator defined by

$$Au = -\mathbb{P}\Delta u$$

The Stokes operator is a self – adjoint positive vectorial operator with a compact inverse. Hence, there exists an othornormal basis of eigenvectors $\{w_n\}$ in L^2_σ, and a sequence of positive eigenvalues

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \to \infty$$

such that

$$Aw_n = \lambda_n w_n, w_n \in D(A).$$

Let $u_n = (u, w_n)$, for $s > 0$, we define the operator A^s by

$$A^s u = \sum_{n=1}^{\infty} \lambda_n^s u_n w_n$$

and the space

$$V^s = \{u \in L^2_\sigma(\Omega):$$

$$u = \sum_{n=1}^{\infty} u_n w_n, \|u\|_s^2 = \sum_{n=1}^{\infty} \lambda_n^s |u_n|^2 < \infty\}.$$

We denote $V = V^1$ and V' its dual.

We recall a trilinear continuous form by setting

$$b(u,v,w) = \sum_{i,j=1}^{3} \int_\Omega u_i(D_j v_j) w_j.$$

This trilinear form is anti – symmetric:

$$b(u,v,w) = -b(u,w,v), u,v,w \in V,$$

in particular, $b(u,v,v) = 0$ for all $u, v \in V$.

Lemma 2.1. *Let* $u: [0,T) \to L^2_\sigma$ *be a weakly continuous weak solution of Navier – Stokes equations on* $[0,T)$*, let*

$$u_K^1 = \sum_{n:\lambda_n \leq K^2} u_n w_n$$

then,

$$|u(t)|^2 + 2\int_0^t \|u\|^2 ds$$

$$= |u_0|^2 + 2 \lim_{K\to\infty} \int_0^t b(u, u_K^1, u) ds$$

for all $0 \leq t < T$.

Proof:

One can see from our assumption that $u_K^1 \in C(0,T;V)$ and $\partial_t u_K^1 \in L^2(0,T;V)$. Thus, using u_K^1 as a test function we obtain

$$|u_K^1|^2 - |u_K^1(0)|^2 + 2\int_0^t \|u_K^1\|^2 ds$$

$$= 2\int_0^t b(u, u_k^1, u) ds.$$

From this we see that the limit of the right hand side exists as $K \to \infty$, which completes the proof of the lemma.

3. Main result

Let $u_K^2 = u - u_K^1$, we have the inequality following:

Let $u \in V^\beta, \beta > \alpha$

$$\|u_K^1\|_\beta^2 = \sum_{n:\lambda_n \le K^2} \lambda_n^\beta |u_n|^2$$

$$= \sum_{n:\lambda_n \le K^2} \lambda_n^{\beta-\alpha} . \lambda_n^\alpha |u_n|^2$$

$$\le K^{2(\beta-\alpha)} . \sum_{n:\lambda_n \le K^2} \lambda_n^\alpha |u_n|^2$$

$$\le K^{2(\beta-\alpha)} . \|u_K^1\|_\alpha^2$$

$$\Rightarrow \|u_K^1\|_\beta \le K^{\beta-\alpha} \|u_K^1\|_\alpha$$

$$\|u_K^2\|_\alpha^2 = \sum_{n:\lambda_n \ge K^2} \lambda_n^\alpha |u_n|^2$$

$$= \sum_{n:\lambda_n > K^2} \lambda_n^{\alpha-\beta} . \lambda_n^\beta |u_n|^2$$

$$\le K^{2(\alpha-\beta)} . \|u_K^2\|_\beta^2$$

$$\Rightarrow \|u_K^2\|_\alpha \le K^{\alpha-\beta} \|u_K^2\|_\beta .$$

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and let u be a weak solution of the Navier – Stokes equations, suppose additionally that*

$$u \in L^3(0, \Gamma; V^s)$$

for some $s \ge \frac{5}{6}$. Then, the energy equality holds

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2 d\tau = \frac{1}{2}\|u_0\|_2^2$$

for all $t \in [0, T)$.

Proof:

In view of Lemma 2.1, it suffices to show that

$$\lim_{K\to\infty} \int_0^t b(u, u_K^1, u) ds = 0$$

to this end let us write

$$b(u, u_K^1, u) = b(u_K^2, u_K^1, u_K^2) + b(u_K^1, u_K^1, u_K^2)$$
$$+ b(u_K^2, u_K^1, u_K^1) + b(u_K^1, u_K^1, u_K^1).$$

The last two terms vanish, so it suffices to estimate only the first two. We use the inequality (see [1])

$$|b(u, v, w)| \le \|u\|_{s_1} . \|v\|_{s_2+1} . \|w\|_{s_3}$$

where $s_1 + s_2 + s_3 \ge \frac{3}{2}$.

We have, for some $s \ge \frac{5}{6}$

$$|b(u_K^2, u_K^1, u_K^2)| \le \|u_K^2\|_{s_1} . \|u_K^1\|_{s_2+1} . \|u_K^2\|_{s_3}$$

$$\le K^{s_1-s} . \|u_K^2\|_s . K^{s_2+1-s} . \|u_K^1\|_s . K^{s_3-s} . \|u_K^2\|_s$$

$$\le K^{s_1+s_2+s_3+1-3s} . \|u_K^2\|_s^2 . \|u_K^1\|_s .$$

We choose $s_1 < s, s_3 < s, s_2 + 1 > s$, and $s_1 + s_2 + s_3 + 1 = 3s$. Hence

$$|b(u_K^2, u_K^1, u_K^2)| \le \|u_K^2\|_s^2 . \|u_K^1\|_s .$$

Which tends to zero as $K \to \infty$. Since in addition,

$$\left|b(u_K^2, u_K^1, u_K^2)\right| \leq \|u\|_s^3 < \infty$$

for all t, by the Dominated Convergence Theorem

$\left|b(u_K^2, u_K^1, u_K^2)\right| \to 0$ as $K \to \infty$

in $L^1(0,T)$. As to the second term, similar estimates

$$\left|b(u_K^1, u_K^1, u_K^2)\right| \leq \left\|u_K^1\right\|_s^2 . \left\|u_K^2\right\|_s$$

which also tends to zero in $L^1(0,T)$ as $K \to \infty$.

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ĐẲNG THỨC NĂNG LƯỢNG CỦA PHƯƠNG TRÌNH NAVIER-STOKES TRONG MIỀN BỊ CHẶN 3 CHIỀU

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Tóm tắt

Đẳng thức về năng lượng

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2 d\tau = \frac{1}{2}\|u_0\|_2^2$$

là một vấn đề mở đối với hệ phương trình Navier – Stokes. Trong bài báo này chúng tôi đưa ra một điều kiện cho đẳng thức năng lượng của nghiệm yếu trong hệ phương trình Navier – Stokes trong không gian ba chiều có miền bị chặn. Chúng tôi chứng minh rằng đẳng thức năng lượng sẽ được giữ nếu nghiệm yếu của phương trình Navier – Stokes thuộc lớp hàm $L^3(0,T;V^s)$ $\left(s \geq \frac{5}{6}\right)$.

Từ khóa: Hệ phương trình Navier – Stokes, nghiệm yếu, đẳng thức năng lượng, bất đẳng thức năng lượng, miền bị chặn.