# NECESSARY OPTIMALITY CONDITIONS FOR FRACTIONAL MULTIOBJECTIVE PROBLEMS VIA CONVEXIFICATORS

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**Abstract:** Many authors studied optimality conditions for vector optimization problems in recent years and obtained Kuhn-Tucker optimality conditions via Lagrange multipliers (see, e.g., Clarke (1983), Kuk, Lee and Tanino (2001), Liang, Huang and Pardalos (2001), Luc (2002),Gong (2010), Luu (2014, 2016), Gadhi (2015),...). Fritz John necessary optimality conditions for fractional multiobjiective optimization problems with equality and inequality constraints which are assumed to be continuous but not necessarily Lipschitz are established. Under a constraint qualification of Mangasarian–Fromovitz type, Kuhn–Tucker necessary optimality conditions for local weak efficient solutions are derived. In this paper, using results of Gadhi (2015), we derive necessary efficiency conditions via convexificators for fractional multi-objective problems with equality and inequality constraints which are assumed to be continuous but not necessarily Lipschitz.

**Keywords:** *convexificator, local weak efficient solution, fractional multiobjiective problem, constraint qualification of Mangasarian-Fromovitz type, Kuhn-Tucker necessary optimality conditions*

## 1 INTRODUCTION

Many authors studied optimality conditions for vector optimization problems in recent years and obtained Kuhn-Tucker optimality conditions via Lagrange multipliers (see, e.g., [1]–[3], [5]–[11]). The notion of nonconvex closed convexificator by Jeyakumar–Luc [4] is a generalization of some notions of known subdifferentials such as the subdifferentials of Clarke [1], Michel-Penot [11]. It has provided good calculus rules for establishing necessary efficiency conditions in nonsmooth optimization. Necessary conditions for efficiency via convexificators can be sharper than those expressed in terms of known subdifferentials as the Clarke, Michel–Penot and Mordukhovich subdifferentials. Luu [6, 7, 8] derived necessary conditions for local weak Pareto and Pareto minima of multiobjective programming problems involving inequality, equality and set constraints in terms of convexificators. Kuk et al. [5] established necessary and sufficient optimality conditions for nonsmooth multiobjective fractional programming problems with inequality constraints via the Clarke subdifferentiabls. Gadhi [2] derived optimality conditions for fractional multiobjective problems involving inequality constraints in terms of convexificators.

The rest of the paper is as follows. Section 2 gives notions and some preliminaries. Section 3 is devoted to derive Fritz John necessary optimality conditions for fractional multiobjective problems via convexificators. Section 4 derives Kuhn–Tucker necessary conditions for this problem under a constraint qualification of Mangasarian Fromovitz type.

## 2 PRELIMINARIES

Given a Banach space $X$, we denote by $X^*$ its topological dual with the canonical dual pairing $\langle .,. \rangle$, and $w^*$ denotes the weak* topology on the dual space. For an extended real valued function $f : X \to \mathbb{R} \cup \{+\infty\}$, the expressions:

$$f_d^-(x;v) = \liminf_{t \to 0} \frac{f(x+tv) - f(x)}{t}, \quad (1)$$

and

$$f_d^+(x;v) = \limsup_{t\to 0} \frac{f(x+tv)-f(x)}{t}. \quad (2)$$

stand for the lower and upper Dini directional derivatives of $f$ at $x$ in the direction of $v$, respectively. Let $C \subset \mathbb{R}^p$ be a pointed ($C \cap -C = \{0\}$) closed convex cone with nonemplty interior which induces a partial order in $\mathbb{R}^p$. Let $A$ be a nonempty subset of $\mathbb{R}^p$. A point $\overline{y} \in A$ is said to be a Pareto (respectively, a weak Pareto) minimal vector of $A$ with respect to $C$ if $A \subset \overline{y} + (\mathbb{R}^p \backslash -C) \cup \{0\}$ (respectively, $A \subset \overline{y} + (\mathbb{R}^p \backslash -int(C))$, where int denotes the topological interior. Consider the following fractional optimization problem $(FMP)$:

$$\begin{cases} \min\left(\frac{f_1(x)}{g_1(x)}, ..., \frac{f_p(x)}{g_p(x)}\right), \\ subject\ to \\ \quad h_j(x) \leqslant 0, j \in I := \{1, ..., m\}, \\ \quad \ell_k(x) = 0, k \in L := \{1, ...r\}. \end{cases}$$

where $f_i, g_i, h_j$ and $\ell_k$ are continous functions defined on $X$ such that $f_i(x) \geqslant 0$ and $g_i(x) > 0$, for all $i, j, k$ and all $x \in X$. Denote by $M$ the feasible set of (FMP)

$$M := \{x \in X : h_j(x) \leq 0, \text{ for all } j = 1, ..., m, \text{ and } \ell_k(x) = 0, \text{ for all } k = 1, ...r\}.$$

For $\overline{x} \in M$, we denote $I(\overline{x}) := \{j \in I : h_j(\overline{x}) = 0\}$. We set $\phi(x) := \left(\frac{f_1(x)}{g_1(x)}, ..., \frac{f_p(x)}{g_p(x)}\right)$. A point $\overline{x} \in M := \{x \in X : h_j(x) \leq 0$ for all $j$, and $\ell_k(x) = 0$ for all $k\}$ is an efficient (resp., weak efficient) solution of $(FMP)$ if $\phi(\overline{x})$ is a Pareto (resp., weak Pareto) minimal vector of $\phi(M)$. A point $\overline{x} \in M$ is a local efficient (resp., weak local efficient) solution of $(FMP)$ if there exists a neighborhood $V$ of $\overline{x}$ such that $\phi(\overline{x})$ is a Pareto (resp., weak Pareto) minimal vector of $\phi(M \cap V)$.

Recall that a point $\overline{x}$ is said to be a regular point in the sense of Ioffe if there exist numbers $K > 0$ and $\delta > 0$ such that for all $x \in X \cap B(\overline{x};\delta)$, $d(x,Q) \leqslant K||\ell(x) - \ell(\overline{x})||$, where $Q = \{x \in X : \ell(x) = \ell(\overline{x})\}, d(x,Q)$ denotes the distance from $x$ to $Q$, $B(\overline{x};\delta)$ stands for the open ball of radius $\delta$ around $\overline{x}$ (see [7]). Note that for Problem (FMP) without equality constraint, a local weak efficient solution of (FMP) is a regular point in the sense of Ioffe.

We recall the following definition in [4].

**Definition 1.** *The funtion $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to have a convexificator $\partial^* f(x)$ at $x$ if $\partial^* f(x) \subset X^*$ is weak* closed, and for each $v \in X$,*

$$f_d^-(x,v) \leqslant \sup_{x^* \in \partial^* f(x)} \langle x^*, v\rangle,$$

*and*

$$f_d^+(x,v) \geqslant \inf_{x^* \in \partial^* f(x)} \langle x^*, v\rangle.$$

Note that convexificators are not necessarily weak* compact or convex. These relaxations allow applications to a large class of nonsmooth continuos funtions. For instance, the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sqrt{x}, & \text{if } x \geqslant 0, \\ -\sqrt{-x}, & \text{if } x < 0, \end{cases}$$

admits noncompact convexificators at 0 of the form $[\alpha, \infty)$ with $\alpha \in \mathbb{R}$ .

**Remark 1.** The convex hull of a convexificator of a locally Lipschitz funtion may be strictly contained in both the Clarke and Michel-Penot subdiffirentials (see [4]). Hence, necessary optimality conditions that are expressed in terms of $\partial^* f(x)$ may provide sharp conditions even for lacally Lipschitz funtions.

**Example 1.**

$$f(x) = \begin{cases} x^2|\cos\frac{\pi}{x}|, & \text{if } x \geqslant 0, \\ 0, & \text{if } x = 0, \end{cases}$$

Then $f_d^+(0;v) = f_d^-(0;v) = 0\,(\forall v \in \mathbb{R})$. The sets $\{0\}$, $[-\pi, \pi]$, and $\{-\pi; \pi\}$ are convexificators of $f$ at $\overline{x} = 0$.

The chain rule for composite functions in [4] is needed in the following.

**Proposition 1.** *Let $f = (f_1, ..., f_n)$ be a continuous function from $X$ to $\mathbb{R}^n$, and let $g$ be continuous funtion from $\mathbb{R}^n$ to $\mathbb{R}$. Suppose that, for each $i = 1, 2, ..., n$, the function $f_i$ admits a bounded convexificator $\partial^* f_i(\overline{x})$ at $\overline{x}$ and that $g$ admits a buonded convexificator $\partial^* g(f(\overline{x}))$ at $f(\overline{x})$. For each $i = 1, ..., n$, if $\partial^* f_i$ is upper semicontinuous at $\overline{x}$ and $\partial^* g$ is upper semicontinuous at $f(\overline{x})$, then the set*

$$\partial^*(g \circ f)(\overline{x}) := \partial^* g(f(\overline{x}))(\partial^* f_1(\overline{x}), ..., \partial^* f_n(\overline{x}))$$

*is a convexificator $g \circ f$ at $\overline{x}$.*

# 3 FRITZ JOHN NECESSARY EFFICIENCY CONDITIONS

In this section, we derive Kuhn-Tucker necessary optimality conditions in terms of convexificators. We also introduce some assumptions for the objective functions and constraints functions of $(FMP)$.

We shall begin with establishing a necessary efficiency condition for (FMP).

**Theorem 1.** *Let $\overline{x} \in E$ be a local weak efficient solution of $(FMP)$. Suppose that $\overline{x}$ is a regular point in the sense of Ioffe. Assume that $h_j$ is continuous and admits a convexificator at $\overline{x}$; $f_i, g_i, \ell_k$ are continuous and admit convexificators $\partial^* f_i(x), \partial^* g_i(x), \partial^* \ell_k(x)$ at $x$ near $\overline{x}$, respectively. They admit bounded convexificators $\partial^* f_i(\overline{x}), \partial^* g_i(\overline{x}), \partial^* \ell_k(\overline{x})$ at $\overline{x}$. Moreover, $\partial^* f_i, \partial^* g_i$ and $\partial^* \ell_k$ are upper semicontinuous at $\overline{x}$. Then there exists vector $(\lambda_1^*, ..., \lambda_p^*) \in \mathbb{R}^p_+ \setminus \{0\}$, $(\mu_0^*, \mu_1^*, ..., \mu_m^*) \in \mathbb{R}^{m+1}_+$, and $(\nu_1^*, ..., \nu_r^*) \in \mathbb{R}^r$ such that*

$$0 \in cl\Big(\mu_0^* \sum_{i=1}^p \lambda_i^* \left(co\, \partial^* f_i(\overline{x}) - \phi_i(\overline{x}) co\, \partial^* g_i(\overline{x})\right)$$
$$+ \sum_{j=1}^m \mu_j^* co\ \partial^* h_j(\overline{x}) + \sum_{k=1}^r \nu_k^* co\ \partial^* \ell_k(\overline{x}),$$

$$\mu_0^* + \sum_{j=1}^m \mu_j^* = 1, \tag{3}$$

$$\mu_j^* h_j(\overline{x}) = 0, \quad j = 1, ..., m, \tag{4}$$

*where cl indicates the weak* clorure.*

*Proof* Since $\overline{x} \in E$ is a local weak efficient solution of $(FMP)$, there exists a neighborhood $U$ of $\overline{x}$ such that

$$\phi(x) - \phi(\overline{x}) \in \mathbb{R}^p \setminus -\mathbb{R}^p_{++}, \tag{5}$$

for all $x \in U \cap E$, where $\mathbb{R}^p_{++} = int\,\mathbb{R}^p_+$.
Let us prove that $\overline{x}$ is a local weak minimal solution of

$$(P_1): \begin{cases} minimize\,(f_1(x) - \phi_1(\overline{x})g_1(x), ..., \\ \qquad\qquad f_p(x) - \phi_p(\overline{x})g_p(x)), \\ subject\ to\ x \in M, \end{cases}$$

where $\phi_i(\overline{x}) = \frac{f_i(\overline{x})}{g_i(\overline{x})}$ $(i = 1, \ldots, p)$.
Assume the contrary, that there exists $x_1 \in U \cap M$ such that

$$(f_i(x_1) - \phi_i(\overline{x})g_i(x_1)) - (f_i(\overline{x}) - \phi_i(\overline{x})g_i(\overline{x}))$$
$$\in -\mathbb{R}_{++}\ (i = 1, \ldots, p).$$

Since $f_i(\overline{x}) - \phi_i(\overline{x})g_i(\overline{x}) = 0$ , one has

$$\frac{f_i(x_1)}{g_i(x_1)} - \frac{f_i(\overline{x})}{g_i(\overline{x})} \in -\mathbb{R}_{++}\,(i = 1, \ldots, p).$$

Hence, $\phi(x_1) - \phi(\overline{x}) \in \mathbb{R}^p_{++}$, which contradicts (4).
Applying the scalarization theorem by Gong ([3], Theorem 3.1) to Problem $(P_1)$ yields the existence of a continuous positively homogeneous subadditive function $\Lambda$ on $X$ satisfying

$$y_2 - y_1 \in \mathbb{R}^p_{++} \Longrightarrow \Lambda(y_1) < \Lambda(y_2),$$

and

$$\Lambda(\varphi(x) - \varphi(\overline{x})) \geqslant 0\ (\forall x \in U \cap M), \tag{6}$$

where $\varphi_i(x) := f_i(x) - \phi_i(\overline{x})g_i(x)$ for all $i \in \{1, ..., p\}, \varphi = (\varphi_1, \ldots, \varphi_p)$. Since $\varphi(\overline{x}) = 0$ and $\Lambda$ is positively homogeneous, it follows from (5) that $\Lambda(\varphi(x)) \geqslant 0 = \Lambda(\varphi(\overline{x}))$, which means that $\overline{x}$ is a local minimal solution of the following scalar problem:

$$(P_2): \begin{cases} \text{minimize } \Lambda(\varphi_1(x) - \varphi_1(\overline{x}), ..., \\ \qquad\qquad \varphi_p(x) - \varphi_p(\overline{x})), \\ \text{subject to, } x \in M. \end{cases}$$

By Theorem 6.4 [7], there exist $\mu_0^* \geqslant 0, \mu_j^* \geqslant 0, j = 1, ..., m, \nu_k^* \in \mathbb{R}, k = 1, ..., r$ such that $\mu_0^* + \sum_{i=1}^{m} \mu_j^* = 1$, and

$$0 \in \mathrm{cl}\Big(\mu_0^* \mathrm{co}\, \partial^* \Lambda(\varphi_1 - \varphi_1(\overline{x}), ..., \varphi_p - \varphi_p(\overline{x}))(\overline{x}) + \sum_{j \in I(\overline{x})} \mu_j^* \mathrm{co}\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* \mathrm{co}\ \partial^* \ell_k(\overline{x})\Big),$$

which is equivalent to the following

$$0 \in \mathrm{cl}\Big(\mu_0^* \mathrm{co}\, \partial^* \Lambda(\varphi_1 - \varphi_1(\overline{x}), ..., \varphi_p - \varphi_p(\overline{x}))(\overline{x}) + \sum_{j=1}^{m} \mu_j^* \mathrm{co}\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* \mathrm{co}\ \partial^* \ell_k(\overline{x}),$$

$$\mu_j^* h_j(\overline{x}) = 0,\ j = 1, \ldots, m.$$

It can be seen that $\Lambda$ is a continuous convex function on $\mathbb{R}^p$, and hence, it is locally Lipschitz on $\mathbb{R}^p$. Hence, we can choose the convex subdifferential $\partial\Lambda(0)$ as a convexificator of $\Lambda$ at 0. Then $\partial\Lambda(0)$ is a bounded convexificator of $\Lambda$ at $(\varphi_1(\overline{x}) - \varphi_1(\overline{x}), \ldots, \varphi_p(\overline{x}) - \varphi_p(\overline{x})) = (0, \ldots, 0)$. Moreover, the mapping $\partial\Lambda$ is upper semicontinuous at $(0, \ldots, 0)$, since $\Lambda$ is locally Lipschitz on $\mathbb{R}^p$. Taking account of Proposition 1 to the composite function $\Lambda(\varphi_1 - \varphi_1(\overline{x}), \ldots, \varphi_p - \varphi_p(\overline{x}))(.)$, one gets that $\partial\Lambda(0)(\partial^*\varphi_1(\overline{x}), ..., \partial^*\varphi_p(\overline{x}))$ is a convexificator of $\Lambda(\varphi_1 - \varphi_1(\overline{x}), \ldots, \varphi_p - \varphi_p(\overline{x}))(.)$ at $\overline{x}$, where $\partial\Lambda(0)$ is the convex subdifferential of $\Lambda$ at 0.

Let see that $\partial^* f_i(\overline{x}) - \phi_i(\overline{x})\partial^* g_i(\overline{x})$ is a convexificator of the function $\varphi_i := f_i - \phi_i(\overline{x}) g_i$ at $\overline{x}$ $(i = 1, \ldots, p)$. In fact, for $i = 1, \ldots, p$, we set $G(u_i, v_i) := u_i - \phi_1(\overline{x}) v_i$, $F(x) := \big(f_i(x), g_i(x)\big)$. Then, $G$ is Fréchet differentiable at $\overline{x}$, and

$$\partial^* G(F(\overline{x})) = \big\{\nabla G(F(\overline{x}))\big\} = \big\{\big(1 - \phi_i(\overline{x})\big)\big\}.$$

Taking account of Proposition 1, it follows that the following set is a convexificator of $G \circ F$ at $\overline{x}$

$$\big(1 - \phi_1(\overline{x})\big)\big((\partial^* f_i(\overline{x}), \partial^* g_i(\overline{x}))\big) = \partial^* f_i(\overline{x}) - \phi_i(\overline{x})\partial^* g_i(\overline{x})).$$

Therefore,

$$0 \in \mathrm{cl}\,\Big(\mu_0^* \partial\Lambda(0)(\mathrm{co}\ \partial^*\varphi_1(\overline{x}), ..., \mathrm{co}\ \partial^*\varphi_p(\overline{x})) + \sum_{j=1}^{m} \mu_j^* \mathrm{co}\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* \mathrm{co}\ \partial^* \ell_k(\overline{x}\Big).$$

Hence, the above inclusion implies that there exists a sequence

$$\theta_n \in \mu_0^* \partial\Lambda(0)(\mathrm{co}\ \partial^*\varphi_1(\overline{x}), ..., \mathrm{co}\ \partial^*\varphi_p(\overline{x})) + \sum_{j=1}^{m} \mu_j^* \mathrm{co}\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* \mathrm{co}\ \partial^* \ell_k(\overline{x})$$

such that $\lim_{n\to\infty} \theta_n = 0$. The latter implies that there exists a sequence $\{\lambda_n\} \subset \partial\Lambda(0)$ such that

$$\theta_n \in \mu_0^* \lambda_n(\mathrm{co}\partial^*\varphi_1(\overline{x}), ..., \mathrm{co}\partial^*\varphi_p(\overline{x})) + \sum_{j=1}^{m} \mu_j^* \mathrm{co}\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* \mathrm{co}\ \partial^* \ell_k(\overline{x}).$$

Since $\partial\Lambda(0)$ is a compact set in $\mathbb{R}^p$, without loss of generality, we can suppose that $\lambda_n \to \lambda^* \in \partial\Lambda(0)$. By letting $n \to \infty$, we get

$$0 \in \mathrm{cl}\,\Big(\mu_0^* \lambda^*(\mathrm{co}\partial^*\varphi_1(\overline{x}), ..., \mathrm{co}\partial^*\varphi_p(\overline{x})) + \sum_{j=1}^{m} \mu_j^* \mathrm{co}\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* \mathrm{co}\ \partial^* \ell_k(\overline{x}).$$

Let us see that $\lambda^* \in \mathbb{R}^p \setminus \{0\}$. In fact, for $y \in \mathbb{R}^p_{++}$, it can be written as $0 - (-y) \in \mathbb{R}^p_{++}$. Therefore,

$$\langle \lambda^*, -y \rangle = \langle \lambda^*, -y - 0 \rangle \leqslant \Lambda(-y) - \Lambda(0) = \Lambda(-y) < \Lambda(0) = 0.$$

Consequently, $\lambda^* \in \mathbb{R}^p_+ \setminus \{0\}$. We have $\lambda^* = (\lambda_1^*, ..., \lambda_p^*) \in \mathbb{R}^p_+ \setminus \{0\}$, and

$$0 \in \mathrm{cl}\Big(\mu_0^*(\lambda_1^*, ..., \lambda_p^*)(\mathrm{co}\partial^*\varphi_1(\overline{x}), ..., \mathrm{co}\partial^*\varphi_p(\overline{x})) + \sum_{j=1}^{m} \mu_j^* \mathrm{co}\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* \mathrm{co}\ \partial^* \ell_k(\overline{x}).$$

Hence,

$$0 \in \mathrm{cl}\Big(\mu_0^* \sum_{i=1}^{p} \lambda_i^* (\mathrm{co}\, \partial^* f_i(\overline{x}) - \phi_i(\overline{x}) \mathrm{co} \partial^* g_i(\overline{x})) + \sum_{j=1}^{m} \mu_j^* \mathrm{co}\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* \mathrm{co}\ \partial^* \ell_k(\overline{x})\Big).$$

The proof is complete. □

# 4 KUHN–TUCKER NECESSARY EFFICIENCY CONDITIONS

To derive Kuhn–Tucker necessary conditions we introduce the following Mangasarian–Fromovitz type constraint qualification $(CQ)$ for multiobjective fractional programming problem $(FMP)$: There exist $v_0 \in X$ and numbers $a_i > 0\, (i \in I(\overline{x}))$ such that

(i) $\langle \xi, v_0 \rangle \leqslant -a_i\ (\forall \xi \in \partial^* h_i(\overline{x}), \forall i \in I(\overline{x}))$;

(ii) $\langle \eta_j, v_0 \rangle = 0\ (\forall \eta_j \in \partial^* \ell(\overline{x}), \forall j \in \{1, \ldots, n\})$.

A Kuhn-Tucker necessary efficiency condition for (FMP) can be stated as follows.

**Theorem 2.** *Let $\overline{x} \in E$ be a local weak efficient solution of $(FMP)$, and let all hypotheses of Theoem 1 be fufilled. Assume furthermore the constraint qualification $(CQ)$ holds. Then, there exists vector $(\alpha_1^*, ..., \alpha_p^*) \in \mathbb{R}_+^p \setminus \{0\}$, $(\mu_0^*, \mu_1^*, ..., \mu_m^*) \in \mathbb{R}_+^{m+1}$, and $(\nu_1^*, ..., \nu_r^*) \in \mathbb{R}^r$ such that*

$$0 \in cl\Big( \sum_{i=1}^{p} \alpha_i^* \,(co\, \partial^* f_i(\overline{x}) - \phi_i(\overline{x}) co\, \partial^* g_i(\overline{x})) + \sum_{j=1}^{m} \mu_j^* co\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* co\ \partial^* \ell_k(\overline{x})\Big),$$

$$\mu_j^* h_j(\overline{x}) = 0, \quad j = 1, ..., m, \tag{7}$$

*Proof* Under Assumptions of Theorem 1, we claim that there exists vector $(\lambda_1^*, ..., \lambda_p^*) \in \mathbb{R}_+^p \setminus \{0\}$, $(\mu_0^*, \mu_1^*, ..., \mu_m^*) \in \mathbb{R}_+^{m+1}$, and $(\nu_1^*, ..., \nu_r^*) \in \mathbb{R}^r$ such that

$$0 \in \mathrm{cl}\Big(\mu_0^* \sum_{i=1}^{p} \lambda_i^* \,(\mathrm{co}\, \partial^* f_i(\overline{x}) - \phi_i(\overline{x}) \mathrm{co}\, \partial^* g_i(\overline{x})) + \sum_{j=1}^{m} \mu_j^* \mathrm{co}\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* \mathrm{co}\ \partial^* \ell_k(\overline{x})\Big),$$

$$\mu_0^* + \sum_{j=1}^{m} \mu_j^* = 1, \tag{8}$$

$$\mu_j^* h_j(\overline{x}) = 0, \quad j = 1, ..., m, \tag{9}$$

If $\lambda_0^* = 0$, then $\sum_{j=1}^{m} \mu_j^* = 1$. Consequently, there exist $\xi_j^{(n)} \in \mathrm{co}\, \partial^* h_j(\overline{x})$, $\eta_k^{(n)} \in \mathrm{co}\, \partial^* \ell_k(\overline{x})$ such that

$$\lim_{n \to \infty} \Big[ \sum_{j \in I(\overline{x})} \mu_j^* \xi_j^{(n)} + \sum_{k \in L} \nu_k^* \eta_k^{(n)} \Big] = 0.$$

Hence,

$$\lim_{n \to \infty} \Big[ \sum_{j \in I(\overline{x})} \mu_j^* \langle \xi_j^{(n)}, v_0 \rangle + \sum_{k \in L} \nu_k^* \langle \eta_k^{(n)}, v_0 \rangle \Big] = 0,$$

where $v_0 \in X$ is available in the constraint qualification (CQ).

On the other hand, since $\mu_i^* \geqslant 0, i = 1, \ldots, m$, $\sum_{j=1}^{m} \mu_j^* = 1$, we have

$$\lim_{n \to \infty} \Big[ \sum_{j \in I(\overline{x})} \mu_j^* \langle \xi_j^{(n)}, v_0 \rangle + \sum_{k \in L} \nu_k^* \langle \eta_k^{(n)}, v_0 \rangle \Big] \leqslant - \sum_{i \in I(\overline{x})} \mu_j^* a_j < 0,$$

which conflicts with (9). We set $\alpha_i^* = \mu_0^* \lambda_i^*$ $(i = 1, \ldots, p)$, we obtain

$$0 \in \mathrm{cl}\Big( \sum_{i=1}^{p} \alpha_i^* \,(\mathrm{co}\, \partial^* f_i(\overline{x}) - \phi_i(\overline{x}) \mathrm{co}\, \partial^* g_i(\overline{x})) + \sum_{j=1}^{p} \mu_j^* \mathrm{co}\ \partial^* h_j(\overline{x}) + \sum_{k=1}^{r} \nu_k^* \mathrm{co}\ \partial^* \ell_k(\overline{x})\Big),$$

with $(\alpha_1^*, \ldots, \alpha_p^*) \neq 0$, which completes the proof. □

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TÓM TẮT

### ĐIỀU KIỆN CẦN TỐI ƯU CHO BÀI TOÁN TỐI ƯU ĐA MỤC TIÊU THƯƠNG QUA DƯỚI VI PHÂN SUY RỘNG

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Trong những năm gần đây, có rất nhiều tác giả nghiên cứu về điều kiện tối ưu cho bài toán cân bằng vectơ trong đó hàm mục tiêu được định nghĩa bởi ánh xạ đơn trị và điều kiện tối ưu đạt được theo quan điểm của nhân tử Lagrange-Kuhn-Tucker (Clarke (1983), Kuk, Lee and Tanino (2001), Liang, Huang and Pardalos (2001), Luc (2002),Gong (2010), Luu (2014, 2016), Gadhi (2015),...). Trong bài báo này, chúng tôi trình bày điều kiện cần Fritz John cho nghiệm hữu hiệu yếu địa phương của bài toán tối ưu đa mục tiêu thương có ràng buộc đẳng thức và bất đẳng thức qua dưới vi phân suy rộng, với giả thiết liên tục mà không cần điều kiện Lipschitz. Dưới điều kiện chính quy kiểu Mangasarian–Fromovitz, điều kiện cần Kuhn–Tucker cho nghiệm hữu hiệu yếu địa phương được thiết lập. Kết quả của bài là mở rộng kết quả của Gadi (2015) cho trường hợp bài toán tối ưu đa mục tiêu thương không có ràng buộc đẳng thức.

**Từ khóa :** *Dưới vi phân suy rộng, nghiệm hữu hiệu yếu địa phương, bà toán tối ưu đa mục tiêu thương, điều kiện chính quy kiểu Mangasarian-Fromovitz, điều kiện cần Kuhn-Tucker.*

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