Fixed point theorem in cone metric spaces

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ABSTRACT

In this paper, we prove new fixed point theorems in complete cone metric spaces via normal cone and give several examples.

Keywords: *Fixed point, Cone metric spaces, Normal cone.*

1 Introduction

In 2007, Huang and Zhang [4] introduced the concept of cone metric space by replacing the set of real numbers by an ordered Banach space and obtained some fixed point theorems for mappings satisfying different contactive conditions. The results in [4] were generalized by Sh. Rezapour and R. Hamlbarani [6] for non-normal cone in a real Banach spaces. Subsequenly, many authors have generalized the results of Huang and Zhang and have studied fixed point theorems for normal and non-normal cone in a real Banach spaces, see for instance [1], [2], [3], [5], [7], [8], [9] and others. In this paper, we prove new fixed point theorems in cone metric space where the uniqueness of the fixed point is not guaranteed.

Let E always be a real Banach spaces with θ is zero vector and P is subset of E. We say that P is a cone in E if

(i) P is closed, nonempty and $P \neq \{\theta\}$,

(ii) $ax + by \in P$ for all $x, y \in P$ and nonnegative real numbers a, b,

(iii) $P \cap (-P) = \{\theta\}$.

For a given cone P in E, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$, while $x \ll y$ will stand for $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. The cone P is called normal cone if there is a number $K > 0$ such that for all $x, y \in E, \theta \preceq x \preceq y$ implies $\|x\| \preceq K\|y\|$. The least positive number satisfying above is called the normal constant of P.

Definition 1.1. (See [4]) Let X be a nonempty set. The mapping $d : X \times X \to E$ is called a cone metric on X if

(d1) $\theta \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = \theta$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \preceq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then (X, d) is called a cone metric space.

Definition 1.2. (See [4]) Let (X, d) is a cone metric space. Let $\{x_n\}$ be a sequence in X. We say that

(i) x is the limit of $\{x_n\}$ if for every $e \in E$ with $\theta \ll e$ there is n_0 such that $d(x_n, x) \ll e$ for all $n \geq n_0$. We denote this by $x_n \to x$ or $\lim\limits_{n\to\infty} x_n = x$.

(ii) $\{x_n\}$ is Cauchy sequence if for every $e \in E$ with $\theta \ll e$ there is n_0 such that $d(x_n, x_m) \ll e$ for all $n, m \geq n_0$.

(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Lemma 1.3. (See [4]) *Let (X, d) is a cone metric space and $\{x_n\}$ be a sequence in X. Then we have:*

(i) If $\{x_n\}$ converges to $x \in X$ then $\{x_n\}$ is a Cauchy sequence.

(ii) If $\{x_n\}$ converges to $x \in X$ and $\{x_n\}$ converges to $y \in X$, then $x = y$.

Lemma 1.4. (See [4]) *Let (X, d) is a cone metric space, P is a normal cone and $\{x_n\}$, $\{y_n\}$ be two sequences in X. Then*

(i) $\lim\limits_{n\to\infty} x_n = x \in X$ if and only if $\lim\limits_{n\to\infty} d(x_n, x) = \theta$.

(ii) $\{x_n\}$ is a Cauchy sequence if and only if $\lim\limits_{n,m\to\infty} d(x_n, x_m) = \theta$.

(iii) If $\lim\limits_{n\to\infty} x_n = x \in X$, $\lim\limits_{n\to\infty} y_n = y \in X$, then

$$\lim_{n\to\infty} d(x_n, y_n) = d(x, y).$$

We can easily prove the following lemma.

Lemma 1.5. *Let P is a normal cone in E. If $\theta \preceq a_n \preceq b_n$ for all $n \geq 1$ and $\lim\limits_{n\to\infty} b_n = \theta$ then $\lim\limits_{n\to\infty} a_n = \theta$.*

2 Fixed point theorem

In this section, we present two fixed point theorems for single valued mappings in complete cone metric spaces and give several examples. Let (X, d) is a cone metric spaces, P be a normal cone with normal constant K and T be a single valued mapping from X to itself. For $\alpha > 0$, we done

$$M(x, y, K, \alpha) = \frac{\|d(x, Ty)\| + \|d(y, Tx)\| + \|d(x, y)\|}{(K+1)\|d(x, Tx)\| + K\|d(y, Ty)\| + K\alpha}.$$

Theorem 2.1. *Let (X, d) is a complete cone metric spaces, P be a normal cone with normal constant K and T be a single valued mapping from X to itself. Suppose there exists $\alpha > 0$ such that*

$$d(Tx, Ty) \preceq M(x, y, K, \alpha).d(x, y),$$

for all $x, y \in X$.
Then

(1) T has at least one fixed point $\bar{x} \in X$;

(2) for any $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point;

(3) if $\bar{x}, \bar{y} \in X$ are two distinct fixed points, then

$$\|d(\bar{x}, \bar{y})\| \geq \frac{K\alpha}{3}.$$

Proof. Let $x_0 \in X$ be a fixed. Consider sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \geq 0$. Set $d_n = d(x_n, x_{n+1})$ for all $n \geq 0$. Then

$$\begin{aligned} d_n &= d(Tx_{n-1}, Tx_n) \\ &\preceq M(x_{n-1}, x_n, K, \alpha).d(x_{n-1}, x_n) \\ &= \frac{(\|d(x_{n-1}, x_{n+1})\| + \|d_{n-1}\|)d_{n-1}}{(K+1)\|d_{n-1}\| + K\|d_n\| + K\alpha}. \qquad (1) \end{aligned}$$

Since

$$d(x_{n-1}, x_{n+1}) \preceq d(x_{n-1}, x_n) + d(x_n, x_{n+1})$$

and by P is a normal with normal constant K, then

$$\begin{aligned} \|d(x_{n-1}, x_{n+1})\| &\leq K\|d(x_{n-1}, x_n) + d(x_n, x_{n+1})\| \\ &\leq K\|d(x_{n-1}, x_n)\| + K\|d(x_n, x_{n+1})\| \\ &= K\|d_{n-1}\| + K\|d_n\|. \qquad (2) \end{aligned}$$

From (1) and (2), we have

$$d_n \preceq \frac{(K+1)\|d_{n-1}\| + K\|d_n\|}{(K+1)\|d_{n-1}\| + K\|d_n\| + K\alpha} d_{n-1}.$$

Set

$$c_n = \frac{(K+1)\|d_{n-1}\| + K\|d_n\|}{(K+1)\|d_{n-1}\| + K\|d_n\| + K\alpha} \text{ for all } n \geq 1.$$

Then $0 \leq c_n < 1$ and $d_n \preceq c_n d_{n-1}$ for all $n \geq 1$. It follows that

$d_n \preceq d_{n-1}$ and $d_n \preceq c_n c_{n-1} ... c_1 d_0$ for all $n \geq 1$.

By the function $f(t) = \frac{t}{t+\alpha}$ is increasing on $[0, +\infty)$, $c_n \leq c_{n-1}$ for all $n \geq 2$. Therefore

$$c_n c_{n-1} ... c_1 \leq c_1^n \to 0 \text{ as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} c_n c_{n-1} ... c_1 = 0 \text{ and } \lim_{n\to\infty} d_n = \theta.$$

On the other hand, for all $n, p \geq 1$, we have

$$\begin{aligned}
&d(x_n, x_{n+p}) \\
&\preceq d(x_n, x_{n+1}) + ... + d(x_{n+p-1}, x_{n+p}) \\
&= d_n + ... + d_{n+p-1} \\
&\preceq c_n c_{n-1} ... c_1 d_0 + ... + c_{n+p-1} c_{n+p-2} ... c_1 d_0 \\
&\preceq p c_n c_{n-1} ... c_1 d_0 \to \theta \text{ as } n \to \infty.
\end{aligned}$$

Thus

$$\lim_{n\to\infty} d(x_n, x_{n+p}) = \theta \text{ for all } p \geq 1.$$

This shows that $\{x_n\}$ be a Cauchy sequence in X. Since X is complete, $\{x_n\}$ converges to some point $\bar{x} \in X$. We claim that $\bar{x}$ is a fixed point of T. Note that

$$\begin{aligned}
&d(x_n, T\bar{x}) = d(Tx_{n-1}, T\bar{x}) \\
&\preceq M(x_{n-1}, \bar{x}, K, \alpha).d(x_{n-1}, \bar{x}) \\
&= \tfrac{\|d(x_{n-1}, T\bar{x})\| + \|d(\bar{x}, x_n)\| + \|d(x_{n-1}, \bar{x})\|}{(K+1)\|d_{n-1}\| + K\|d(\bar{x}, T\bar{x})\| + \alpha}.d(x_{n-1}, \bar{x})
\end{aligned}$$

$\to \theta$ as $n \to \infty$. We get

$$\lim_{n\to\infty} x_n = T\bar{x}.$$

Hence, $T\bar{x} = \bar{x}$ holds, thus, $\bar{x}$ is a fixed point of T. If $\bar{y}$ is a fixed point of T with $\bar{x} \neq \bar{y}$, then

$$\begin{aligned}
d(\bar{x}, \bar{y}) &= d(T\bar{x}, T\bar{y}) \\
&\preceq M(\bar{x}, \bar{y}, K, \alpha).d(\bar{x}, \bar{y}) \\
&= \frac{3\|d(\bar{x}, \bar{y})\|}{\alpha}.d(\bar{x}, \bar{y}).
\end{aligned}$$

This implies

$$\|d(\bar{x}, \bar{y})\| \geq \frac{\alpha}{3}.$$

□

Remark 2.2. Note that in Theorem 2.1, the ration $M(x, y, K, \alpha)$ might be greater than 1 and the uniqueness of the fixed point is not guaranteed. The following example shows this note precisely.

Example 2.3. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ a normal cone with normal constant $K = 1$ in E. Let $X = \{0, 1, 2\}$ and let $d : X \times X \to \mathbb{R}^2$ by

$$d(0,0) = d(1,1) = d(2,2) = (0,0),$$
$$d(0,1) = d(1,0) = (\frac{3}{2}, 0),$$
$$d(0,2) = d(2,0) = (1,0),$$
$$d(1,2) = d(2,1) = (2,0).$$

Then (X, d) is a complete cone metric space. Let $T : X \to X$ by $T0 = 0, T1 = 1$ and $T2 = 2$. For $\alpha = 1$, we have

$$M(0,1,1,1) = \frac{9}{2}, M(1,0,1,1) = \frac{9}{2},$$
$$M(1,2,1,1) = \frac{4}{3}, M(2,1,1,1) = \frac{4}{3},$$
$$M(0,2,1,1) = \frac{5}{3}, M(2,0,1,1) = \frac{5}{3}.$$

Then

$$\begin{aligned}
d(T0, T1) = d(0,1) &\preceq M(0,1,1,1).d(0,1), \\
d(T1, T0) = d(1,0) &\preceq M(1,0,1,1).d(1,0), \\
d(T1, T2) = d(1,2) &\preceq M(1,2,1,1).d(1,2), \\
d(T2, T1) = d(2,1) &\preceq M(2,1,1,1).d(2,1), \\
d(T0, T2) = d(0,2) &\preceq M(0,2,1,1).d(0,2), \\
d(T2, T0) = d(2,0) &\preceq M(2,0,1,1).d(2,0).
\end{aligned}$$

Therefore T satisfies all the conditions of Theorem 2.1 for $\alpha = 1$. Also, T has three distinct fixed points $\{0, 1, 2\}$ and

$$\|d(\bar{x}, \bar{y})\| \geq \frac{\alpha}{3} = \frac{1}{3} \text{ for all } \bar{x} \neq \bar{y} \in X.$$

Corollary 2.4. *Let (X, d) is a complete cone metric spaces, P be a normal cone with normal constant K and T be a single valued mapping from X to itself. Suppose there exists $\alpha > 0$ such that*

$$d(Tx, Ty) \preceq M(x, y, K, \alpha).d(x, y),$$

for all $x, y \in X$. Then T has a unique fixed point if $M(x, y, K, \alpha) < 1$ for all $x, y \in X$.

Proof. From Theorem 2.1, T has a fixed point $\bar{x}$. If $\bar{y}$ is a fixed point of T, then

$$d(\bar{x}, \bar{y}) = d(T\bar{x}, T\bar{y}) \preceq M(\bar{x}, \bar{y}, K, \alpha).d(\bar{x}, \bar{y}).$$

This implies

$$[1 - M(\bar{x}, \bar{y}, K, \alpha)]d(\bar{x}, \bar{y}) \preceq \theta.$$

Since $M(\bar{x}, \bar{y}, K, \alpha) < 1$, $d(\bar{x}, \bar{y}) = \theta$. Hence $\bar{x} = \bar{y}$. □

Example 2.5. Let $E = \mathbb{R}^2$, $P = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ a normal cone with normal constant $K = 1$ in E. Let $X = \{0, 1, 2\}$ and let $d : X \times X \to \mathbb{R}^2$ by

$$d(0, 0) = d(1, 1) = d(2, 2) = (0, 0),$$
$$d(0, 1) = d(1, 0) = (\frac{1}{2}, 0),$$
$$d(0, 2) = d(2, 0) = (1, 0),$$
$$d(1, 2) = d(2, 1) = (\frac{1}{2}, 0).$$

Then (X, d) is a complete cone metric space. Let $T : X \to X$ by $T0 = 0, T1 = 0$ and $T2 = 0$. For $\alpha = 2$, we have

$$M(0, 0, 1, 2) = 0, M(1, 1, 1, 2) = \frac{2}{7},$$
$$M(2, 2, 1, 2) = \frac{2}{5}, M(0, 1, 1, 2) = \frac{2}{5},$$
$$M(1, 0, 1, 2) = \frac{1}{3}, M(1, 2, 1, 2) = \frac{2}{5},$$
$$M(2, 1, 1, 2) = \frac{4}{9}, M(0, 2, 1, 2) = \frac{2}{3},$$
$$M(2, 0, 1, 2) = \frac{1}{2}.$$

Then $M(x, y, 1, 2) < 1$ for all $x, y \in X$. Moreover, since $d(Tx, Ty) = \theta$ for all $x, y \in X$, then

$$d(Tx, Ty) \preceq M(x, y, 1, 2).d(x, y), \text{ for all } x, y \in X.$$

Therefore T satisfies all the conditions of Corollary 2.4 for $\alpha = 2$. Also, T has a unique fixed points $\bar{x} = 0$.

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Summary

Định lý điểm bất động trong không gian metric nón

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Trong bài báo này, chúng tôi chứng minh định lý điểm bất động trong không gian metric nón với nón chuẩn tắc và một số ví dụ tính toán minh họa cho kết quả trên.

Key words and phrases: *Điểm bất động, Không gian metric nón, Nón chuẩn tắc.*

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