NON COHEN-MACAULAY IN DIMENSION MORE THAN S LOCUS

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Abstract. Let $(R, \mathfrak{m})$ be a Noetherian local ring, M a finitely generated R-module. Let $s \geq -1$ be an integer. Following N. Zamani [14], M is a Cohen-Macaulay module in dimension more than s if every s.o.p. of M is an M-sequence in dimension more than s. In this paper, we introduce the notions of non Cohen-Macaulay in dimension more than s locus, $\mathrm{nCM}_{>s}(M)$ and i-th pseudo support in dimension more than s of M, $\mathrm{Psupp}^i_{>s}(M)$. It is clear that a Cohen-Macaulay module in dimension more than s for $s = -1$ is Cohen-Macaulay module. In this case, the non Cohen-Macaulay in dimension more than s locus and i-th pseudo support in dimension more than s are exactly the non Cohen-Macaulay and i-th pseudo support of M. Next, we give a description $\mathrm{nCM}_{>s}(M)$ via these i-th pseudo supports in dimension more than s of M.

Key words: *Cohen-Macaulay modules in dimension more than s, non Cohen-Macaulay in dimension more than s locus, the i-th pseudo support, Noetherian dimension, quotient of a Cohen-Macaulay local ring.*

1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module with $\dim M = d$. It is well known that the Cohen-Macaulay modules play an important role in the theory of Noetherian rings and finitely generated modules. Recall that M is called *Cohen-Macaulay* if every system of parameters (s.o.p. for short) of M is an M-sequence. There are some extensions of the concepts of M-sequence and Cohen-Macaulay modules, among which are the notions of M-sequence in dimension more than s introduced by Brodmann-Nhan [1] and Cohen-Macaulay modules in dimension more than s defined by Zamani [14]. Let $s \geq -1$ be an integer. A sequence $(x_1, \ldots, x_r)$ of elements in $\mathfrak{m}$ is said to be an *M-sequence in dimension more than s* if $x_i \notin \mathfrak{p}$, for all $\mathfrak{p} \in \mathrm{Ass}_R(M/(x_1, \ldots, x_{i-1})M)$ satisfying $\dim(R/\mathfrak{p}) > s$, for all $i = 1, \ldots, r$. We say that M is a *Cohen-Macaulay module in dimension more than s* if every s.o.p. of M is an M-sequence in dimension more than s.

It is clear that M-sequences in dimension more than s for $s = -1, 0, 1$ are exactly M-sequences, f-sequences with respect to M in sense of Cuong-Schenzel-Trung [6], and generalized regular sequences with respect to M in sense of Nhan [10], respectively. Therefore, Cohen-Macaulay modules in dimension more than s for $s = -1, 0, 1$ respectively are Cohen-Macaulay modules, f-modules defined in [6] and generalized f-modules introduced in Nhan-Morales [11].

Non Cohen-Macaulay locus of M, denoted by $\mathrm{nCM}(M)$, is defined by

$$\mathrm{nCM}(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid M_{\mathfrak{p}} \text{ is not Cohen-Macaulay }\}.$$

Let $i \geq 0$ be an integer. Following M. Brodmann and R. Y. Sharp [2], *the i-th pseudo support of M*, denoted by $\mathrm{Psupp}^i_R(M)$, is defined by

$$\mathrm{Psupp}^i_R(M) = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p} R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0\}.$$

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In 2010, N. T. Cuong, L. T. Nhan and N. T. K. Nga (see [5]) used pseudo support to describe the non-Cohen-Macaulay locus of M as follows

$$\mathrm{nCM}(M) = \bigcup_{0\leqslant i<j\leqslant d} (\mathrm{Psupp}^i_R(M) \cap \mathrm{Psupp}^j_R(M)).$$

Similar to $\mathrm{nCM}(M)$ and pseudo support $\mathrm{Psupp}^i_R(M)$, we denote $\mathrm{nCM}_{>s}(M)$ and $\mathrm{Psupp}^i_{>s}(M)$ are non Cohen-Macaulay in dimension more than s locus and the i-th pseudo support in dimension more than s of M, respectively. The aim of this paper is to describe the non Cohen-Macaulay in dimension more than s locus via the i-th pseudo support in dimension more than s of M. Firstly, we give the following definition.

Definition 1.1. (a) *Non Cohen-Macaulay in dimension more than s locus of M,* denoted by $\mathrm{nCM}_{>s}(M)$, is defined by

$$\mathrm{nCM}_{>s}(M) = \{\mathfrak{p} \in \operatorname{Spec} R \mid M_{\mathfrak{p}} \text{ is not Cohen-Macaulay in dimension more than s}\}.$$

(b) For an integer $i \geq 0$, *the i-th pseudo support in dimension more than s of M,* denoted by $\mathrm{Psupp}^i_{>s}(M)$, is defined by

$$\mathrm{Psupp}^i_{>s}(M) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \text{N-dim}_{R_{\mathfrak{p}}} \big(H^{i-\dim R/\mathfrak{p}}_{\mathfrak{p} R_{\mathfrak{p}}}(M_{\mathfrak{p}})\big) > s\}.$$

Note that if $s = -1$, then non Cohen-Macaulay in dimension more than -1 locus is non Cohen-Macaulay locus and the i-th pseudo support in dimension more than -1 of M is i-th pseudo support of M. So, we have the description in the case $s = -1$. For $s \geq 0$ is an integer, we have the following theorem, which is the main result of this paper.

Theorem 1.2.

$$\mathrm{nCM}_{>s}(M) \subseteq \bigcup_{1\leqslant i<j\leqslant d} (\mathrm{Psupp}^i_{>s}(M) \cap \mathrm{Psupp}^j_{>s}(M)).$$

The converse statement holds true when R is a quotient of a Cohen-Macaulay local ring. Furthermore, if M is equidimensional then

$$\mathrm{nCM}_{>s}(M) = \bigcup_{1\leqslant i<d} \mathrm{Psupp}^i_{>s}(M).$$

2. Proof of Theorem 1.2

To prove this theorem, we need some following lemmas. Firstly, we recall the notion of N-dim by using the terminology of Kirby [9].

Definition 2.1. *Noetherian dimension of an Artinian R-module A*, denoted by $\text{N-dim}_R A$, is defined inductively as follows: when $A = 0$, put $\text{N-dim}_R A = -1$. Then by induction, for an integer $d \geq 0$, we put $\text{N-dim}_R A = d$ if $\text{N-dim}_R A < d$ is false and for every ascending sequence $A_0 \subseteq A_1 \subseteq ...$ of submodules of A, there exists n_0 such that $\text{N-dim}_R(A_{n+1}/A_n) < d$ for all $n > n_0$. Therefore $\text{N-dim}_R A = 0$ if and only if A is a non-zero Noetherian module. In this case, A has a finite length.

Remark 2.2. Suppose that $(R, \mathfrak{m})$ is local and $A \neq 0$.
(i) Kirby has shown that $\ell_R(0 :_A \mathfrak{m}^n)$ is a polynomial with rational coefficients when $n \gg 0$. Roberts [13] has then proved that

$$\text{N-dim}_R A = \deg \ell_R(0 :_A \mathfrak{m}^n) = \inf\{t \geq 0 : \exists x_1, ..., x_t \in \mathfrak{m} : \ell_R(0 :_A (x_1, ..., x_t)R) < \infty\}.$$

(ii) Let $\widehat{R}$ be the $\mathfrak{m}$-adic completion of R. Then A has a natural stucture as an $\widehat{R}$-module as follows: let $(x_n) \in \widehat{R}$, where $x_n \in R$, and let $u \in A$. Then we get $u.\mathfrak{m}^n = 0$ for $n \gg 0$. Therefore $x_n.u$ is constant for $n \gg 0$. So we defined $(x_n).u = x_n.u$ for $n \gg 0$. With this structure, a subset of A is an R-submodule if and only if it is an $\widehat{R}$-submodule. Therefore we have $\text{N-dim}_R A = \text{N-dim}_{\widehat{R}} A$.

Lemma 2.3. ([8, Theorem 3.7]) ([3, Lemma 3.1]). *Let $0 \leq s < d$. If $\text{N-dim}_R(H^i_{\mathfrak{m}}(M)) \leq s$ for all $i < d$ then M is Cohen-Macaulay in dimension more than s. The converse is also true if R is a quotient of a Cohen-Macaulay local ring.*

Lemma 2.4. ([4, Corollary 3.2, 3.6]) .

$$\text{N-dim}_R(H^i_{\mathfrak{m}}(M)) \leq i, \textit{ for all } i.$$

In particular, $\text{N-dim}_R(H^d_{\mathfrak{m}}(M)) = d$.

Proof of Theorem 1.2. Let $\mathfrak{p} \in \text{nCM}_{>s}(M)$. Then $M_\mathfrak{p}$ is not Cohen-Macaulay in dimension more than s. By Lemma 2.3, there exist $1 \leqslant t < \dim M_\mathfrak{p}$ such that $\text{N-dim}_{R_\mathfrak{p}}\left(H^t_{\mathfrak{p} R_\mathfrak{p}}(M_\mathfrak{p})\right)$ $> s$. Set $i = t + \dim(R/\mathfrak{p})$, we have

$$\text{N-dim}_{R_\mathfrak{p}}\left(H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\right) > s.$$

Hence $\mathfrak{p} \in \text{Psupp}^i_{>s}(M)$. Set $k = \dim M_\mathfrak{p}$. Then $t < k$. Since $M_\mathfrak{p}$ is not Cohen-Macaulay in dimension more than s, we have $k > s$. Therefore, it follows by Lemma 2.4 that

$$\text{N-dim}_{R_\mathfrak{p}}(H^k_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})) = k > s.$$

Set $j = k + \dim(R/\mathfrak{p})$. Then $\text{N-dim}_{R_\mathfrak{p}}\left(H^{j-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\right) > s$. This implies $\mathfrak{p} \in \text{Psupp}^j_{>s}(M)$. So,

$$\mathfrak{p} \in \text{Psupp}^i_{>s}(M) \cap \text{Psupp}^j_{>s}(M).$$

Since $1 \leq t < k = \dim M_\mathfrak{p}$, we have $1 \leq i < j \leq d$. Then

$$\text{nCM}_{>s}(M) \subseteq \bigcup_{1 \leqslant i < j \leqslant d} (\text{Psupp}^i_{>s}(M) \cap \text{Psupp}^j_{>s}(M)).$$

Conversely, let $\mathfrak{p} \in \text{Psupp}^i_{>s}(M) \cap \text{Psupp}^j_{>s}(M)$, $1 \leq i < j \leq d$. We have

$$\text{N-dim}_{R_\mathfrak{p}}\left(H^{j-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\right) > s \text{ and } \text{N-dim}_{R_\mathfrak{p}}\left(H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\right) > s.$$

It is clear that $j - \dim(R/\mathfrak{p}) \leq \dim M_\mathfrak{p}$. Then there exsist $t = j - \dim(R/\mathfrak{p}) < \dim M_\mathfrak{p}$ satisfy

$$\text{N-dim}_{R_\mathfrak{p}}\left(H^t_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\right) > s.$$

Since R is a quotient of a Cohen-Macaulay local ring, it follows by Lemma 2.3 that $M_{\mathfrak{p}}$ is not Cohen-Macaulay in dimension more than s. Therefore, $\mathfrak{p} \in \mathrm{nCM}_{>s}(M)$. This proves

$$\bigcup_{1 \leqslant i < j \leqslant d} (\mathrm{Psupp}^i_{>s}(M) \cap \mathrm{Psupp}^j_{>s}(M)) \subseteq \mathrm{nCM}_{>s}(M).$$

Now assume that M is equidimensional then $\mathrm{Psupp}^i_{>s}(M) \subseteq \mathrm{Psupp}^d_{>s}(M)$, for all $i < d$. In fact, for any $\mathfrak{p} \in \mathrm{Psupp}^i_{>s}(M)$, we have $\text{N-dim}_{R_{\mathfrak{p}}}\left(H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})\right) > s$. Since R is catenary and M is equidimensional then $\dim M_{\mathfrak{p}} = d - \dim R/\mathfrak{p}$. Therefore, we have by lemma 2.4 that

$$\begin{aligned}\text{N-dim}_{R_{\mathfrak{p}}}\left(H^{d-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})\right) &= d - \dim R/\mathfrak{p} \geq i - \dim R/\mathfrak{p} \\ &\geq \text{N-dim}_{R_{\mathfrak{p}}}\left(H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})\right) > s.\end{aligned}$$

So, $\mathfrak{p} \in \mathrm{Psupp}^d_{>s}(M)$. This implies

$$\begin{aligned}\mathrm{nCM}_{>s}(M) &= \bigcup_{1 \leqslant i < j \leqslant d} (\mathrm{Psupp}^i_{>s}(M) \cap \mathrm{Psupp}^j_{>s}(M)) \\ &= \bigcup_{1 \leqslant i < d} (\mathrm{Psupp}^i_{>s}(M).\end{aligned}$$

Thus, the assertion is proved. $\square$

Note that when R is not a quotient of a Cohen-Macaulay local ring, the equality in Theorem 1.2 is not true. Here is an example.

Example 2.5. Let $(R, \mathfrak{m})$ be the Noetherian local domain of dimension 2 constructed by Ferrand and Raynaud in [7] such that the $\mathfrak{m}$-adic completion $\widehat{R}$ of R has an associated prime $\widehat{q}$ of dimension 1. Then R is not a quotient of a Cohen-Macaulay local ring. Consider the case $s = 0$, we have R is Cohen-Macaulay in dimension more than 0. So, $\mathrm{nCM}_{>0}(R) = \emptyset$. Since $\text{N-dim}_{R_{\mathfrak{p}}}\left(H^0_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})\right) = 0$, we have

$$\begin{aligned}\mathrm{Psupp}^1_{>0}(R) &= \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \text{N-dim}_{R_{\mathfrak{p}}}\left(H^{1-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})\right) > 0\} \\ &= \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \dim R/\mathfrak{p} = 0\} = \{\mathfrak{m}\}.\end{aligned}$$

On the other hand, for any $\mathfrak{p} \in \mathrm{Spec}\, R$, $\dim R/\mathfrak{p} = 1$ we have $\dim R_{\mathfrak{p}} = \dim R - \dim R/\mathfrak{p} = 1$. Hence $\text{N-dim}_{R_{\mathfrak{p}}}\left(H^1_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})\right) = 1$. Then

$$\begin{aligned}\mathrm{Psupp}^2_{>0}(R) &= \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \text{N-dim}_{R_{\mathfrak{p}}}\left(H^{2-\dim R/\mathfrak{p}}_{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})\right) > 0\} \\ &= \{\mathfrak{m}\} \cup \{\mathfrak{p} \in \mathrm{Spec}\, R \mid \dim R/\mathfrak{p} = 1\}.\end{aligned}$$

Therefore $\mathrm{nCM}_{>0}(R) \subsetneq (\mathrm{Psupp}^1_{>0}(R) \cap \mathrm{Psupp}^2_{>0}(R))$.

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TÓM TẮT

QUỸ TÍCH KHÔNG COHEN-MACAULAY CHIỀU LỚN HƠN S

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Cho $(R, \mathfrak{m})$ là một vành địa phương Noether, M là một R-môđun hữu hạn sinh. Cho $s \geq -1$ là một số nguyên. Theo N. Zamani [14], M là môđun Cohen-Macaulay chiều lớn hơn s nếu mọi hệ tham số của M là M-dãy chính quy chiều lớn hơn s. Trong bài báo này, chúng tôi giới thiệu các khái niệm quỹ tích không Cohen-Macaulay chiều lớn hơn s, ký hiệu $\mathrm{nCM}_{>s}(M)$ và tập giả giá thứ i chiều lớn hơn s của M, ký hiệu $\mathrm{Psupp}^i_{>s}(M)$. Dễ thấy rằng môđun Cohen-Macaulay chiều lớn hơn s với $s = -1$ chính là môđun Cohen-Macaulay. Trong trường hợp này, quỹ tích không Cohen-Macaulay chiều lớn hơn s và tập giả giá thứ i chiều lớn hơn s chính là quỹ tích không Cohen-Macaulay và tập giả giá thứ i của M. Tiếp theo, chúng tôi đưa ra mô tả quỹ tích $\mathrm{nCM}_{>s}(M)$ qua các tập giả giá thứ i chiều lớn hơn s của M.

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Từ khóa: *Môđun Cohen-Macaulay chiều lớn hơn s, quỹ tích không Cohen-Macaulay chiều lớn hơn s, giả giá thứ i, chiều Noether, thương của vành Cohen-Macaulay địa phương.*

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