FINITE-DIMENSIONAL APPROXIMATION FOR SYSTEM OF NONLINEAR MONOTONE ILL-POSED EQUATIONS IN BANACH SPACE

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SUMMARY

In fact have many problem lead to solving the system of nonlinear monotone ill-posed equations such as image reconstruction problem, signal recovery problem, optimal control problem ... These problems are studied in finite-dimensional spaces as well as infinite-dimensional spaces. In this paper we study the convergence and convergence rate for regularization solutions in connection with the finite-dimensional approximation for system of nonlinear monotone ill-posed equations in Banach spaces.

Keywords: *Ill-posed problem, regularization method, monotone, hemi-continuous, inverse-strongly monotone.*

INTRODUCTION

Let E be a real reflexive Banach space and E^* be its dual space, which both are assumed to be strictly convex. For the sake of simplicity, norms of E and E^* are denoted by the symbol $\|.\|$ and $\langle x^*, x\rangle$ denotes the value of the linear and continuous functional $x^* \in E^*$ at the point $x \in E$.

In this paper, we consider the problem of finding a solution for a system of the following equations

$$A_i(x) = \theta, \quad i = 0, 1, \dots, N, \tag{1}$$

where N is a fixed positive iteger, A_0 is a *hemi-continuous* and monotone mapping, the other mappings A_i are λ_i-inverse strongly monotone with domain $D(A_i) = E$, $i = 0, 1, \dots, N$.

Denote by S_i the set of solutions for ith equation in (1). Throughout this paper, we assume that $S := \cap_{i=0}^{N} S_i \neq \emptyset$.

It is well-known in [1] that each equation in (1), in general, is ill-posed, by this we mean that the solutions do not depend continuously on the data f_i. Consequently, the system of equations (1), in general, is ill-posed. Therefore strong convergence and stability of approximate solutions can be proved only by applying some regularization procedure. In any method of finding $z \in S$, the main aim is to establish a continuous dependence of approximate solutions on data perturbation.

To solve (1), in 2006 Buong [5] presented the regularization method of Browder-Tikhonov type when A_i, $i = 0, 1, \dots, N$ are *hemi-continuous*, monotone and potential mappings with $D(A_i) = E$, in [10] this result was modified for the case A_0 is a Lipsschitz continuous and monotone mapping and the others A_i are λ_i-inverse strongly monotone mapping in Hilbert spaces. Further, in [12] Thuy presented the regularization method when A_i, $i = 0, 1, \dots, N$ are inverse strongly monotone mappings in Banach spaces.

In this paper, a regularezed solution for (1) is defined by the following equation

$$A_0(x) + \alpha^{\mu} \sum_{i=1}^{N} A_i(x) + \alpha J^s(x) = \theta, \tag{2}$$

with a regularization parameter $\alpha > 0$, a fixed

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number $\mu \in (0,1)$, an initial point $x^+ \notin S$, and a generalized duality mapping J^s of E, i.e., $J^s : E \to E^*$ that satisfies the condition

$$\langle U^s(x), x\rangle = \|x\|^s, \quad \|U^s(x)\|\|x\|^{s-1}, \quad s \geq 2.$$

Furthermore, we have the following property of the J^s (see [1])

$$\langle J^s(x) - J^s(y), x - y\rangle \geq m_s\|x - y\|^s, \quad \forall\, x, y \in E,\ m_s > 0,\ s \geq 2. \tag{3}$$

In what follows, we collect some definitions on monotone operators and their useful properties. We refer the reader [1] for more details.

Definition 1. *A mapping A of domain $\mathcal{D}(A) \subset E$ into E^* is called Lipschitz continuous with a constant $L > 0$ if*

$$\|A(x) - A(y)\| \leq L\|x - y\| \quad \forall\, x, y \in \mathcal{D}(A).$$

Definition 2. *A mapping A of domain $\mathcal{D}(A) \subset E$ into E^* is called*

(i) *monotone if*

$$\langle A(x) - A(y), x - y\rangle \geq 0 \quad \forall\, x, y \in \mathcal{D}(A);$$

(ii) *λ-inverse strongly monotone if there exists a positive constant λ such that*

$$\langle A(x) - A(y), x - y\rangle \geq \lambda\|A(x) - A(y)\|^2,$$

for all $x, y \in \mathcal{D}(A)$.

Definition 3. *A mapping A of domain $\mathcal{D}(A) \subset E$ into E^* is called hemi-continuous at a point $x_0 \in \mathcal{D}(A)$ if $A(x_0 + tx) \rightharpoonup Ax_0$ as $t \to 0$ for any x such that $x_0 + tx \in \mathcal{D}(A)$;*

If A is *hemi*-continuous at every point of $\mathcal{D}(A)$, then A is said to be *hemi*-continuous.

Obviously, any λ-inverse strongly monotone mapping A is monotone and L-Lipschitz continuous with constant $L = 1/\lambda$. And any monotone and *hemi*-continuous mapping A with $\mathcal{D}(A) = E$ is maximal monotone.

Definition 4. *A reflexive Banach space E is said to be a* **E**-*space if it is strictly convex and has the Kadec-Klee property: for any sequence $\{x_n\}$, the weak convergence $x_n \rightharpoonup x$ and convergence of norms $\|x_n\| \to \|x\|$ imply strong convergence $x_n \to x$.*

MAIN RESULTS

Lemma 1. *Let E be a reflexive Banach space, E^* be a strictly convex Banach space, $J^s : \mathcal{D}(J^s) = E \to E^*$ be a normalized duality mapping. Suppose that $A_0 : \mathcal{D}(A_0) = E \to E^*$ is monotone and hemicontinuous, the other mappings $A_i : \mathcal{D}(A_i) = E \to E^*$, $i = 1, \dots, N$, are λ_i-inverse strongly monotone. Then, for each $\alpha > 0$ equation (2) has a unique solution x_α.*

Proof.

Clearly, for each fixed $\alpha > 0$, the mapping

$$A^N(\cdot) := \alpha^\mu \sum_{i=1}^{N} A_i(\cdot),$$

is monotone and $(\alpha^\mu \sum_{i=1}^N L_i)$-Lipschitz continuous with $\mathcal{D}(A^N) = E$, where $L_i = 1/\lambda_i$. So, A^N is *hemi*-continuous. Consequently, the mapping $A = A_0 + A^N$ is monotone and *hemi*-continuous with $\mathcal{D}(A) = E$. Hence, A is maximally monotone (see [1]). Furthermore, under our assumptions, we have that J^s is *demi*-continuous and single-valued, and $\mathcal{D}(J^s) = E$. Therefore, Theorem 1.7.4 (see [1]) ensures the solvability of equation (2). On the other hand, because J^s is strictly monotone, the mapping $A + \alpha J^s$ is also strictly monotone. Thus, equation (2) has a unique solution x_α for each $\alpha > 0$. □

In computation, the finite-dimensional approximation for (2) is the important problem. As usualy, it can be approximated by the following equation:

$$A_0^n(x) + \alpha^\mu \sum_{i=1}^{N} A_i^n(x) + \alpha J^{sn}(x) = \theta, \tag{4}$$

where $A_i^n = P_n^* A_i P_n$, $J^{sn} = P_n^* J^s P_n$, $\alpha > 0$, $x \in E_n$, $P_n : E \to E_n$ is linear projection operator from E onto the finite dimensional subspace E_n of E, $P_n^* : E^* \to E_n^*$ is conjugate operator to P_n, and

$$E_n \subset E_{n+1}, \quad \forall n, \quad P_n x \to x, \quad \forall x \in E.$$

Without loss of generality, we suppose that $\|P_n\| = 1$. As also for (2), this equation has a unique solution $x_{\alpha,n}$ for all $\alpha > 0$ and n.

Theorem 1. *The sequence $\{x_{\alpha,n}\}$ of solutions of the equation (4) converges to a solution x_α of (2), as $n \to \infty$.*

Proof. It follows from (4) that

$$\begin{aligned}&\langle A_0^n(x_{\alpha,n}), x_{\alpha,n} - P_n x_\alpha\rangle \\ &+ \alpha^\mu \sum_{i=1}^N \langle A_i^n(x_{\alpha,n}), x_{\alpha,n} - P_n x_\alpha\rangle \\ &+ \alpha\langle J^{sn}(x_{\alpha,n}), x_{\alpha,n} - P_n x_\alpha\rangle = 0.\end{aligned} \tag{5}$$

By using (3), we have

$$\begin{aligned}&\alpha m_s \|x_{\alpha,n} - P_n x_\alpha\|^s \\ &\le \alpha\langle J^s(x_{\alpha,n}) - J^s(P_n x_\alpha), x_{\alpha,n} - P_n x_\alpha\rangle \\ &= \alpha\langle J^{sn}(x_{\alpha,n}) - J^{sn}(P_n x_\alpha), x_{\alpha,n} - P_n x_\alpha\rangle.\end{aligned}$$

From (5), we have

$$\begin{aligned}\alpha m_s \|x_{\alpha,n} - P_n x_\alpha\|^s &\le \langle A_0^n(x_{\alpha,n}), P_n x_\alpha - x_{\alpha,n}\rangle \\ &+ \alpha^\mu \sum_{i=1}^N \langle A_i^n(x_{\alpha,n}), P_n x_\alpha - x_{\alpha,n}\rangle \\ &+ \alpha\langle J^{sn}(P_n x_\alpha), P_n x_\alpha - x_{\alpha,n}\rangle.\end{aligned} \tag{6}$$

Since $A_i^n = P_n^* A_i P_n$, $J^{sn} = P_n^* J^s P_n$, it follows from (6) that

$$\begin{aligned}\alpha m_s \|x_{\alpha,n} - P_n x_\alpha\|^s &\le \langle A_0(x_{\alpha,n}), P_n x_\alpha - x_{\alpha,n}\rangle \\ &+ \alpha^\mu \sum_{i=1}^N \langle A_i(x_{\alpha,n}), P_n x_\alpha - x_{\alpha,n}\rangle \\ &+ \alpha\langle J^s(P_n x_\alpha), P_n x_\alpha - x_{\alpha,n}\rangle.\end{aligned} \tag{7}$$

Using the monotonicity of A_i, it follows from (7) that

$$\begin{aligned}\alpha m_s \|x_{\alpha,n} - P_n x_\alpha\|^s &\le \langle A_0(P_n x_\alpha), P_n x_\alpha - x_{\alpha,n}\rangle \\ &+ \alpha^\mu \sum_{i=1}^N \langle A_i(P_n x_\alpha), P_n x_\alpha - x_{\alpha,n}\rangle \\ &+ \alpha\langle J^s(P_n x_\alpha), P_n x_\alpha - x_{\alpha,n}\rangle.\end{aligned} \tag{8}$$

Which leads to the following inequality

$$\begin{aligned}&\alpha m_s \|x_{\alpha,n} - P_n x_\alpha\|^s \\ &\le \left[\|A_0(P_n x_\alpha)\| + \alpha^\mu \sum_{i=1}^N \|A_i(P_n x_\alpha)\|\right] \times \\ &\times \|P_n x_\alpha - x_{\alpha,n}\| + \alpha\|P_n x_\alpha\|\|P_n x_\alpha - x_{\alpha,n}\|.\end{aligned} \tag{9}$$

This implies that the sequence $\{x_{\alpha,n}\}$ is bounded. Without loss of generality, we suppose that $\{x_{\alpha,n}\}$ is convergent weakly to $\overline{x}_\alpha$. Since $A_i^n = P_n^* A_i P_n$, $J^{sn} = P_n^* J^s P_n$, the monotonicity of A_i and J^s, it follows from (4) that

$$\begin{aligned}&\left\langle A_0(x^n) + \alpha^\mu \sum_{i=1}^N A_i(x^n), x^n - x_{\alpha,n}\right\rangle \\ &\qquad + \langle \alpha J^s(x^n), x^n - x_{\alpha,n}\rangle \ge 0,\end{aligned}$$

with $\alpha > 0$, $x^n = P_n x \in E_n$.

By letting $n \to \infty$ in this inequality, using the property of A_i, P_n and $x_{\alpha,n} \rightharpoonup \overline{x}_\alpha$, for all $x \in E$ we have

$$\left\langle A_0(x) + \alpha^\mu \sum_{i=1}^N A_i(x) + \alpha J^s(x), x - \bar{x}_\alpha\right\rangle \ge 0.$$

Since, (2) has a unique solution, it follows that $\overline{x}_\alpha = x_\alpha$ and sequence $\{x_{\alpha,n}\}$ converges weakly to x_α. From (8) deduce the sequence $\{x_{\alpha,n}\}$ converges strongly to x_α, as $n \to \infty$.

□

Let $\gamma_n(z) = \|(I - P_n)(z)\|$, $z \in S$, where I denotes the identity operator in E.

Theorem 2. *Let E, E^*, J^s, S, and A_i, $(i = 0, \dots, N)$ be as in Lemma 1. Suppose that E is an **E**-space. If $\gamma_n(z)/\alpha \to 0$ as $\alpha \to 0$ and $n \to \infty$, then the sequence $\{x_{\alpha,n}\}$ converges to $x^0 \in S$.*

Proof. For $z \in S$, $z^n = P_n z$, it follows from (4) that

$$\begin{aligned} &\langle A_0^n(x_{\alpha,n}), x_{\alpha,n} - z^n\rangle \\ &+ \alpha^\mu \Big\langle \sum_{i=1}^N A_i^n(x_{\alpha,n}), x_{\alpha,n} - z^n \Big\rangle \\ &+ \alpha\langle J^{sn}(x_{\alpha,n}), x_{\alpha,n} - z^n\rangle = 0, \end{aligned} \tag{10}$$

where $x_{\alpha,n}$ is solution of (4). It follows from (10), $A_i^n = P_n^* A_i P_n$, $J^{sn} = P_n^* J^s P_n$, $P_n P_n = P_n$, and the monotonicity of A_i that

$$\begin{aligned} &\alpha\langle J^s(x_{\alpha,n}), x_{\alpha,n} - z^n\rangle = \alpha\langle J^{sn}(x_{\alpha,n}), x_{\alpha,n} - z^n\rangle \\ &= \Big\langle A_0^n(x_{\alpha,n}) + \alpha^\mu \sum_{i=1}^N A_i^n(x_{\alpha,n}), z^n - x_{\alpha,n} \Big\rangle \\ &\le \Big\langle A_0(z^n) + \alpha^\mu \sum_{i=1}^N A_i(z^n), z^n - x_{\alpha,n} \Big\rangle. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\alpha\langle J^s(x_{\alpha,n}), x_{\alpha,n} - z^n\rangle \\ &\le \langle A_0(z^n) - A_0(z), z^n - x_{\alpha,n}\rangle \\ &\quad + \alpha^\mu \Big\langle \sum_{i=1}^N \big(A_i(z^n) - A_i(z)\big), z^n - x_{\alpha,n} \Big\rangle \\ &\le \|A_0(z^n) - A_0(z)\| \|z^n - x_{\alpha,n}\| \\ &\quad + \alpha^\mu \Big[\sum_{i=1}^N \|A_i(z^n) - A_i(z)\| \Big] \|z^n - x_{\alpha,n}\|. \end{aligned} \tag{11}$$

On the other hand, by using

$$\|A_i(z^n) - A_i(z)\| \le \bar{K}\gamma_n(z), \tag{12}$$

where $\bar{K}$ is some positive constant depending only on z, it follows from (4) that

$$\begin{aligned} \langle J^s(x_{\alpha,n}), x_{\alpha,n} - z^n\rangle \le &\alpha(1 + N\alpha^\mu)\times \\ &\times \|z^n - x_{\alpha,n}\|. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\langle J^s(x_{\alpha,n}), x_{\alpha,n}\rangle - \langle J^s(x_{\alpha,n}), z^n\rangle \\ &\le \frac{\bar{K}\gamma_n(z)}{\alpha}(1 + N\alpha^\mu)(\|x_{\alpha,n}\| + \|z^n\|). \end{aligned} \tag{13}$$

Thus, we have

$$\|x_{\alpha,n}\|^2 - \|x_{\alpha,n}\| \Big[\|z\| + \frac{\bar{c}(\alpha)}{\alpha} \Big] - \frac{\bar{c}(\alpha)}{\alpha}\|z\| \le 0,$$

where $\bar{c}(\alpha) = K\gamma_n(z)(1+N\alpha^\mu)$. Consequently, we have

$$\begin{aligned} \|x_{\alpha,n}\| \le &\frac{1}{2}\Big(\|z\| + \frac{\bar{c}(\alpha)}{\alpha} \Big) \\ &+ \frac{1}{2}\sqrt{\Big(\|z\| + \frac{\bar{c}(\alpha)}{\alpha} \Big)^2 + \frac{4\bar{c}(\alpha)}{\alpha}\|z\|} \\ \le &\|z\| + \frac{\bar{c}(\alpha)}{\alpha} + \sqrt{\frac{\bar{c}(\alpha)}{\alpha}\|z\|}. \end{aligned} \tag{14}$$

Since $\gamma_n(z)/\alpha \to 0$ as $\alpha \to 0$ and $n \to \infty$, it means that $\{x_{\alpha,n}\}$ is bounded. Since E is reflexive, there exists a subsequence of $\{x_{\alpha,n}\}$, that converges weakly to $\overline{x} \in E$. For the sake of simplicity, assume that $x_{\alpha,n} \rightharpoonup \overline{x}$ as $\alpha \to 0$ and $n \to \infty$. First, we prove that $\overline{x} \in S_0$. Indeed, by virtue of $A_i^n = P_n^* A_i P_n$, $J^{sn} = P_n^* J^s P_n$, the monotonicity of A_i and J^s, it follows from (4) that

$$\begin{aligned} &\langle A_0^n(P_n x), P_n x - x_{\alpha,n}\rangle \\ &= \langle A_0^n(P_n x) - A_0^n(x_{\alpha,n}) + A_0^n(x_{\alpha,n}), P_n x - x_{\alpha,n}\rangle \\ &\ge \langle A_0(x_{\alpha,n}), P_n x - x_{\alpha,n}\rangle \\ &= \alpha^\mu \sum_{i=1}^N \langle A_i(x_{\alpha,n}), x_{\alpha,n} - P_n x\rangle \\ &\quad + \alpha\langle J^s(x_{\alpha,n}), x_{\alpha,n} - P_n x\rangle \\ &\ge \alpha^\mu \sum_{i=1}^N \langle A_i(P_n x), x_{\alpha,n} - P_n x\rangle \\ &\quad + \alpha\langle J^s(P_n x), x_{\alpha,n} - P_n x\rangle, \quad \forall x \in E. \end{aligned} \tag{15}$$

Since $P_n P_n = P_n$, the last inequality has form

$$\begin{aligned} &\langle A_0(P_n x), P_n x - x_{\alpha,n}\rangle \\ &\ge \alpha^\mu \sum_{i=1}^N \langle A_i(P_n x), x_{\alpha,n} - P_n x\rangle \\ &\quad + \alpha\langle J^s(P_n x), x_{\alpha,n} - P_n x\rangle, \quad \forall x \in E. \end{aligned} \tag{16}$$

After tending $\alpha \to 0$, and $n \to \infty$ in this inequality, we obtain

$$\langle A_0(x), x - \overline{x}\rangle \ge 0, \quad \forall x \in E.$$

Thus, $\overline{x} \in S_0$ (see [14]). Now, we shall prove that $\overline{x} \in S_i$, $i = 1, 2, \ldots, N$. Again, from (4), the monotonicity of A_i, J^s, and (12), we have

$$\begin{aligned}
&\sum_{i=1}^{N} \langle A_i(x_{\alpha,n}) - A_i(P_n z), x_{\alpha,n} - P_n z\rangle \\
&= \sum_{i=1}^{N} \langle A_i^n(x_{\alpha,n}) - A_i^n(P_n z), x_{\alpha,n} - P_n z\rangle \\
&= \sum_{i=1}^{N} \langle A_i^n(x_{\alpha,n}) - A_i^n(P_n z), x_{\alpha,n} - P_n z\rangle \\
&= \sum_{i=1}^{N} \langle -A_i^n(P_n z), x_{\alpha,n} - P_n z\rangle \\
&\quad + \alpha^{1-\mu}\langle J^{sn}(x_{\alpha,n}), P_n z - x_{\alpha,n}\rangle \\
&\quad + \frac{1}{\alpha^{\mu}}\langle -A_0^n(x_{\alpha,n}), x_{\alpha,n} - P_n z\rangle \\
&\le \sum_{i=1}^{N} \langle -A_i(P_n z) + A_i(z), x_{\alpha,n} - P_n z\rangle \\
&\quad + \alpha^{1-\mu}\langle J^{s}(P_n z), P_n z - x_{\alpha,n}\rangle \\
&\quad + \frac{1}{\alpha^{\mu}}\langle -A_0(P_n z) + A_0(z), x_{\alpha,n} - P_n z\rangle \\
&\le \left[\sum_{i=1}^{N} \|A_i(z) - A_i(P_n z)\|\right] \|x_{\alpha,n} - P_n z\| \\
&\quad + \alpha^{1-\mu}\langle J^{s}(P_n z), P_n z - x_{\alpha,n}\rangle \\
&\quad + \frac{1}{\alpha^{\mu}}\Big[\|A_0(z) - A_0(P_n z)\|\Big] \|x_{\alpha,n} - P_n z\| \\
&\le \frac{1}{\alpha^{\mu}}\Big[N\alpha^{\mu} + \bar{K}\gamma_n(z) + \bar{K}\gamma_n(z)N\alpha^{\mu}\Big]\times \\
&\qquad \|x_{\alpha,n} - P_n z\| \\
&\quad + \alpha^{1-\mu}\langle J^{s}(P_n z), P_n z - x_{\alpha,n}\rangle, z \in S.
\end{aligned}$$

Which together with the λ_i-inverse strongly monotone property of A_i implies

$$\begin{aligned}
&\sum_{i=1}^{N} \lambda_i \|A_i(x_{\alpha,n}) - A_i(P_n z)\|^2 \\
&\le \sum_{i=1}^{N} \langle A_i(x_{\alpha,n}) - A_i(P_n z), x_{\alpha,n} - P_n z\rangle \\
&\le \Big[\frac{1+N}{\alpha^{\mu}} + \frac{\gamma_n(z)}{\alpha}\alpha^{1-\mu}(\bar{K} + N\bar{K}\alpha^{\mu})\Big]\times \\
&\qquad \|x_{\alpha,n} - P_n z\| \\
&\quad + \alpha^{1-\mu}\|P_n(z)\|\|P_n z - x_{\alpha,n}\|, z \in S.
\end{aligned}$$

Thus, $\|A_i(x_{\alpha,n}) - A_i(z)\| \to 0$ as $\alpha \to 0$ and $n \to \infty$ with $\gamma_n(z)/\alpha \to 0$. Note that, each mapping A_i is maximal monotone (see [3], Theorem 1.3, p.40). As we know that (see [1], Lemma 1.4.5, p.39), the graph $G(A)$ of any maximal monotone mapping A from a reflexive Banach space E to E^* is demiclosed, that is, $x_n \to x$, $y_n \rightharpoonup f$ or $x_n \rightharpoonup x$, $y_n \to f$, where $(x_n, y_n) \in G(A)$, imply that $(x, f) \in G(A)$. Thus, $A_i(\bar{x}) = \theta$, $i = 1, 2, \ldots, N$, that is, $\bar{x} \in S_i$. Next, since each S_i is closed convex, S is also closed convex. Therefore, the element x^0 in S with minimal norm in the strictly convex Banach space E is unique. And now, from (14) with z replaced by $\bar{x}$, it implies that $\|x_{\alpha,n}\| \to \|\bar{x}\|$ and $\|\bar{x}\| \le \|z\|$, for all $z \in S$. Hence, $x_{\alpha,n} \to \overline{x}$ (because E is an **E**-space), which is the element x^0, that we have to find.

□

The next theorem will be give the convergence for regularization solutions in connection with the finite-dimensional approximation, and estimate convergence rate for $\{x_{\alpha,n}\}$ under the conditions:

$$\begin{aligned}\|A_0(y) - f_0 - [A_0'(x^0)]^*(y - x^0)\| \\ \le \tau\|A_0(y) - f_0\|,\end{aligned} \tag{17}$$

for y in some neighborhood of $x^0 \in S$, where $A_0'(x^0)$ denotes the derivative of A_0 at x^0, $[A_0'(x^0)]^*$ is the adjoint of $A_0'(x^0)$, and τ is some positive constant.

The condition (17) is called *tangential cone condition* and is widely used in the analysis of regularization methods for solving nonlinear ill-posed inverse problems (see [7]).

Theorem 3. *Assume that the following conditions hold:*

(i) *A_0 is continuously Fréchet differentiable with (17) for $x = x^0$, and the other each A_i is L_i-Lipschitz continuous in some neighbourhood of x^0;*

(ii) *there exists an element* $\omega \in E$ *such that*

$$[A_0'(x^0)]^*\omega = J^s(x^0),$$

where J^s *satisfies condition* (3);

(iii) *the parameter* α *is chosen by* $\alpha \sim \gamma_n^\nu$, $0 < \nu < 1$, *where* $\gamma_n = \max_{x\in S}\gamma_n(x)$.

Then,

$$\|x_{\alpha,n} - x^0\| = O(\gamma_n^h + \gamma_n^l),$$

where

$$h = \min\left\{\frac{1-\nu}{s-1}, \frac{\mu\nu}{s}\right\}, l = \min\left\{\frac{1}{s}, \frac{\nu}{s-1}\right\},$$

and $s \geq 2$.

Proof. Replacing $P_n x_\alpha$ by $x_n^0 = P_n x^0$ in (8), we obtain

$$\begin{aligned}\alpha m_s\|x_{\alpha,n} - x_n^0\|^s &\leq \langle A_0(x_n^0), x_n^0 - x_{\alpha,n}\rangle \\ &+ \alpha^\mu\sum_{i=1}^N\langle A_i(x_n^0), x_n^0 - x_{\alpha,n}\rangle \\ &+ \alpha\langle J^s(x_n^0), x_n^0 - x_{\alpha,n}\rangle.\end{aligned} \tag{18}$$

We have

$$\begin{aligned}&\langle A_0(x_n^0), x_n^0 - x_{\alpha,n}\rangle \\ &\quad\leq \|A_0(x_n^0) - A_0(x^0)\|\|x_n^0 - x_{\alpha,n}\| \\ &\quad\leq \widetilde{C_0}\gamma_n\|x_n^0 - x_{\alpha,n}\|,\end{aligned} \tag{19}$$

where $\widetilde{C_0}$ is a positive constant depending only on x^0. And also, we have

$$\begin{aligned}&\sum_{i=1}^N\langle A_i(x_n^0), x_n^0 - x_{\alpha,n}\rangle \\ &\leq \Big[\sum_{i=1}^N\|A_i(x_n^0) - A_i(x^0)\|\Big]\|x_n^0 - x_{\alpha,n}\| \\ &\leq \Big(\sum_{i=1}^N\widetilde{C_i}\gamma_n + N\Big)\|x_n^0 - x_{\alpha,n}\| \\ &\leq \Big(\widetilde{C}\gamma_n + N\Big)\|x_n^0 - x_{\alpha,n}\|,\end{aligned} \tag{20}$$

where $\widetilde{C_i}$ is a positive constant depending only on x^0 and $\widetilde{C} = \sum_{i=1}^N\widetilde{C_i}$, and

$$\begin{aligned}&\langle J^s(x_n^0) - J^s(x^0), x_n^0 - x_{\alpha,n}\rangle \\ &\leq C(\widetilde{R})\gamma_n^\nu\|x_n^0 - x_{\alpha,n}\|,\ 0 < \nu < 1,\end{aligned} \tag{21}$$

where $\widetilde{R} > \|x^0\|$ and

$$\begin{aligned}&\langle J^s(x^0), x_n^0 - x_{\alpha,n}\rangle \\ &= \langle J^s(x^0), x_n^0 - x^0\rangle + \langle J^s(x^0), x^0 - x_{\alpha,n}\rangle \\ &= \langle J^s(x^0), x_n^0 - x^0\rangle + \langle\omega, A_0'(x^0)(x^0 - x_{\alpha,n})\rangle \\ &\leq \|J^s(x^0)\|\|x_n^0 - x^0\| + \|\omega\|\|A_0'(x^0)(x^0 - x_{\alpha,n})\| \\ &\leq \|x^0\|\|(I - P_n)x^0\| + \|\omega\|\|A_0'(x^0)(x^0 - x_{\alpha,n})\| \\ &\leq \widetilde{R}\gamma_n + \|\omega\|(\tau+1)\|A_0(x_{\alpha,n})\|,\end{aligned} \tag{22}$$

and

$$\begin{aligned}&\|A_0(x_{\alpha,n})\| \leq \alpha^\mu\sum_{i=1}^N\|A_i(x_{\alpha,n})\| + \alpha\|x_{\alpha,n}\| \\ &\leq \alpha^\mu\sum_{i=1}^N\Big(\|A_i(x_{\alpha,n} - A_i(x^0)\|\Big) + \alpha\|x_{\alpha,n}\| \\ &\leq \alpha^\mu\sum_{i=1}^N L_i\|x_{\alpha,n} - x_n^0\| + \alpha^\mu\widetilde{C}\gamma_n + \alpha^\mu N + \alpha\|x_{\alpha,n}\|.\end{aligned} \tag{23}$$

Thus, we have

$$\begin{aligned}&\alpha m_s\|x_{\alpha,n} - x_n^0\|^s \\ &\leq \widetilde{C_0}\gamma_n\|x_n^0 - x_{\alpha,n}\| + \alpha^\mu\Big(\widetilde{C}\gamma_n + N\Big)\|x_n^0 - x_{\alpha,n}\| \\ &\quad + \alpha C(\widetilde{R})\gamma_n^\nu\|x_n^0 - x_{\alpha,n}\| + \alpha\widetilde{R}\gamma_n \\ &\quad + \alpha\|\omega\|(\tau+1)\Big[(1+\alpha^\mu N) + \alpha^\mu\sum_{i=1}^N L_i\|x_{\alpha,n} - x_n^0\| \\ &\qquad + \alpha^\mu\widetilde{C}\gamma_n + \alpha\|x_{\alpha,n}\|\Big] \\ &\leq \Big[\widetilde{C_0}\gamma_n + \alpha^\mu(\widetilde{C}\gamma_n + N) + \alpha C(\widetilde{R})\gamma_n^\nu \\ &\qquad + \alpha^{\mu+1}\|\omega\|(\tau+1)\sum_{i=1}^N L_i\Big] \\ &\quad \times\|x_n^0 - x_{\alpha,n}\| + \alpha\widetilde{R}\gamma_n + \alpha\|\omega\|(\tau+1)(1+N\alpha^\mu) \\ &\quad + \alpha^{\mu+1}\|\omega\|(\tau+1)\widetilde{C}\gamma_n + \alpha^2\|\omega\|(\tau+1)\|x_{\alpha,n}\|.\end{aligned} \tag{24}$$

The last inequality implies that

$$
\begin{aligned}
&m_s\|x_{\alpha,n} - x_n^0\|^s \\
&\leq \Big[\frac{\widetilde{C_0}\gamma_n}{\alpha} + \alpha^\mu \frac{\widetilde{C}\gamma_n + N}{\alpha} + C(\widetilde{R})\gamma_n^\nu \\
&\qquad + \alpha^\mu\|\omega\|(\tau+1)\sum_{i=1}^N L_i\Big]\|x_n^0 - x_{\alpha,n}\| \\
&\quad + \widetilde{R}\gamma_n + \|\omega\|(\tau+1)(1+N\alpha^\mu) \\
&\quad + \alpha^\mu\|\omega\|(\tau+1)\widetilde{C}\gamma_n + \alpha\|\omega\|(\tau+1)\|x_{\alpha,n}\|.
\end{aligned} \tag{25}
$$

If α is chosen by condition (iii), then $\alpha \leq 1$, from (25) we have

$$
\begin{aligned}
&\|x_{\alpha,n} - x_n^0\|^s \\
&\leq \Big[C_1\gamma_n^{1-\nu} + C_2\gamma_n^{\mu\nu} + C_3\gamma_n^\nu\Big]\|x_n^0 - x_{\alpha,n}\| \\
&\quad + C_4\gamma_n + C_5\gamma_n^{\mu\nu},
\end{aligned} \tag{26}
$$

where C_i, $i = 1, \ldots, 5$ are some positive constants. Using the implication

$$
\begin{aligned}
&a, b, c \geq 0, \quad s > t, \\
&\quad a^s \leq ba^t + c \Rightarrow a^s = \mathrm{O}(b^{s/(s-t)} + c),
\end{aligned}
$$

we obtain

$$\|x_{\alpha,n} - x_n^0\| = \mathrm{O}(\gamma_n^h + \gamma_n^l).$$

Thus

$$\|x_{\alpha,n} - x^0\| = \mathrm{O}(\gamma_n^h + \gamma_n^l).$$

□

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TÓM TẮT

XẤP XỈ HỮU HẠN CHIỀU CHO HỆ PHƯƠNG TRÌNH PHI TUYẾN ĐƠN ĐIỆU ĐẶT KHÔNG CHỈNH TRONG KHÔNG GIAN BANACH

Trần Thị Hương

Trường Cao đẳng Kinh tế - Kỹ thuật, Đại học Thái Nguyên

Trong thực tế có rất nhiều bài toán được đưa về bài toán tìm nghiệm của hệ phương trình phi tuyến đơn điệu đặt không chỉnh như: bài toán khôi phục ảnh, bài toán khôi phục tín hiệu, bài toán điều khiển tối ưu . . . Những bài toán này đã được nghiên cứu trong các không gian hữu hạn chiều cũng như vô hạn chiều. Trong bài báo này, tác giả nghiên cứu sự hội tụ và tốc độ hội tụ của nghiệm hiệu chỉnh đã được xấp xỉ hữu hạn chiều cho hệ phương trình phi tuyến đơn điệu đặt không chỉnh trong không gian Banach.

Từ khóa: *Bài toán đặt không chỉnh, phương pháp hiệu chỉnh, đơn điệu, hemi-liên tục, ngược đơn điệu mạnh.*

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