COMPLETENESS AND FIXED POINTS IN STRONG b− METRIC SPACE

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ABSTRACT

In 1968, Kannan proves that the following result: *Let* (X, d) *be a complete metric space and T be a self-mapping on X satisfying*

$$d(Tx, Ty) \leq r\{d(x, Tx) + d(y, Ty)\}$$

for all $x, y \in X$ *and* $r \in (0, 12)$*. Then, T has a unique fixed point* $\bar{x} \in X$ *and for any* $x \in X$*, the sequence of iterates* $\{T nx\}$ *converges to* $\bar{x}$. The mapping satisfying the above contraction condition is called Kannan mapping. Another important meaning of the Kannan mapping is being able to describe the completeness of space in terms of the fixed-point property of the mapping. This was proved by Subrahmanyam in 1975. Means, a metric space (X, d) is complete if and only if every Kannan mapping has a unique fixed point in X. In this paper, we consider the same problem in the case of strong b-metric space as generalization of result of Subrahmanyam.

TÍNH ĐẦY ĐỦ VÀ ĐỊNH LÝ ĐIỂM BẤT ĐỘNG TRONG KHÔNG GIAN b− METRIC MẠNH

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TỪ KHÓA

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Dãy Cauchy
Ánh xạ Kannan
Không gian b-metric mạnh
Không gian b-metric mạnh đầy đủ

TÓM TẮT

Năm 1968, Kannan đã chứng minh kết quả sau: Cho (X, d) là không gian metric đầy đủ và T là ánh xạ đi từ X vào chính nó thỏa mãn

$$d(Tx, Ty) \leq r\{d(x, Tx) + d(y, Ty)\}$$

với mọi $x, y \in X$ *và* $r \in (0, 12)$*. Khi đó, T có duy nhất điểm bất động* $\bar{x} \in X$ *và với mỗi* $x \in X$*, dãy lặp* $\{T nx\}$ *hội tụ tới* $\bar{x}$. Ánh xạ thỏa mãn điều kiện co trên được gọi là ánh xạ Kannan. Một ý nghĩa quan trọng khác của ánh xạ Kannan là có thể mô tả tính đầy đủ của không gian với tính chất điểm bất động của ánh xạ. Điều này đã được chứng minh bởi Subrahmanyam vào năm 1975. Có nghĩa là, một không gian metric (X, d) là đầy đủ nếu và chỉ nếu mọi ánh xạ Kannan có một điểm bất động duy nhất trong X. Trong bài báo này, chúng tôi xem xét vấn đề tương tự trong trường hợp không gian b−metric mạnh là mở rộng kết quả của Subrahmanyam.

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1 Introduction

It is well-known that the study of fixed point theorem and characteristic of metric space is important in applied mathematics and there are many research of mathematicians. For example see [1, 2, 3, 4] and others. In 1968, Kannan proved the following result.

Theorem 1. ([5])*Let (X, d) be a complete metric space and T be a self-mapping on X satisfying*

$$d(Tx, Ty) \leq r\{d(x, Tx) + d(y, Ty)\} \tag{1}$$

for all $x, y \in X$ and $r \in (0, \frac{1}{2})$. Then, T has a unique fixed point $\bar{x} \in X$ and for any $x \in X$, the sequence of iterates $\{T^n x\}$ converges to $\bar{x}$.

The mapping T satisfying the condition of the above theorem is called Kannan mapping. Theorem 1 shows that in a complete metric space, any Kannan map always has a unique fixed point. And in 1972, Chatterjea proved:

Theorem 2. ([6])*Let (X, d) be a complete metric space and T be a self-mapping on X satisfying*

$$d(Tx, Ty) \leq r\{d(x, Ty) + d(y, Tx)\} \tag{2}$$

for all $x, y \in X$ and $r \in [0, \frac{1}{2})$. Then, T has a fixed point $\bar{x} \in X$.

The mapping satisfying the condition (2) of Theorem 2 is called Chatterjea mapping. And in 1975, Subrahmanyam confirmed that

Theorem 3. ([7]) *A metric space (X, d) in which every mappings $T : X \to X$ satisfying the conditions*

(i) there exists $\alpha > 0$ such that $d(Tx, Ty) < \alpha\big(d(x, Tx) + d(y, Ty)\big)$ for all $x, y \in X$;

(ii) $T(X)$ is countable;
has a fixed point, is complete.

The condition (i) in this theorem is related to condition (1) in Theorem 1 and (2) in Theorem 2. As it is remarked in [7] Theorem 2 provides completeness of metric spaces on which every Kannan map, or every Chatterjea map, has a fixed point. In 2014, Kirk and Shahzad [8] introduced strong b-metric space as a sharp generalization of metric space. Our idea here is to study a complete characteristic of strong b-metric space related to fixed points. First of all, we introduce some concepts of strong b-metric space.

Definition 1. ([8]) Let X be a nonempty set and $K \geq 1$ be a real number. A function $d : X \times X \to [0; +\infty)$ is called a strong b-metric on X if

(D1) $d(x, y) = 0$ if and only if $x = y$;

(D2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(D3) $d(x, y) \leqslant d(x, z) + Kd(z, y)$ for all $x, y, z \in X$.
Then (X, d, K) is called a strong b-metric space.

Next we consider the convergent of sequences in strong b-metric space.

Definition 2. ([8]) Let (X, d, K) be a strong b-metric spase and $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) The sequence $\{x_n\}$ is called convergent to x if $\lim_{n\to\infty} d(x_n, x) = 0$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(ii) The sequence $\{x_n\}$ is called Cauchy sequence in X if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

(iii) The strong b-metric space (X, d, K) is called complete if every Cauchy sequence in X is convergent.

2 Main results

To prove our results we need some proposition.

Proposition 1 ([8])**.** *Let (X, d, K) be a strong b-metric space and $\{x_n\}$ be a sequence in X. Then*

(1) *If $\{x_n\}$ converges to $x \in X$ and $\{x_n\}$ converges to $y \in X$, then $x = y$.*

(2) *If $\lim_{n\to\infty} x_n = x \in X$ and $\lim_{n\to\infty} y_n = y \in X$, then $\lim_{n\to\infty} d(x_n, y_n) = d(x, y)$.*

Proposition 2 ([8])**.** *Let (X, d, K) be a strong b-metric space, let $\{x_n\}$ be a sequence in X and suppose*

$$\sum_{i=1}^{\infty} d(x_i, x_{i+1}) < +\infty.$$

Then $\{x_n\}$ is a Cauchy sequence.

Our main results is following:

Theorem 4. *Let (X, d, K) be a complete strong b-metric space and T be a self-mapping on X satisfying*

$$d(Tx, Ty) \leqslant r\{d(x, Tx) + d(y, Ty)\} \tag{3}$$

for all $x, y \in X$ and $r \in [0, \frac{1}{2})$. Then, T has a unique fixed point $x^* \in X$ and for any $x \in X$, the sequence of iterates $\{T^n x\}$ converges to x^*.

Proof. Let $x_0 \in X$ be arbitrary element in X, we define a sequence $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for all $n \geq 0$. Set $d_n = d(x_n, x_{n+1})$ for any $n \geq 0$. From hypothesis, we have

$$\begin{aligned} d_{n+1} &= d(x_{n+1}, x_{n+2}) \\ &= d(Tx_n, Tx_{n+1}) \\ &\leqslant r\{d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})\} \\ &= r\{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\} \\ &= r\{d_n + d_{n+1}\}. \end{aligned}$$

So we have

$$d_{n+1} \leqslant \frac{r}{1-r} d_n.$$

This implies that

$$d_n \leq q^n d_0$$

for all $n \in \mathbb{N}^*$, where $q = \frac{r}{1-r}$ is a nonnegative and $q < 1$. So we get

$$\sum_{n=1}^{\infty} d_n \leqslant d_0 \sum_{n=1}^{\infty} q^n < +\infty.$$

It implies that $\{x_n\}$ is a Cauchy sequence in X by Proposition 2. Since X is complete strong b-metric space, there exists $x^* \in X$ such that $\lim_{n\to\infty} x_n = x^*$. Next we will show that x^* is a fixed point of T. Indeed, from the condition $d_n \leqslant q^n d_0$ we have $d_n \to 0$ as $n \to \infty$. So from hypothesis, we have

$$d(x_{n+1}, Tx^*) \leqslant r(d(x_n, Tx_n) + d(x^*, Tx^*).$$

Let $n \to \infty$ we have

$$d(x^*, Tx^*) \leqslant rd(x^*, Tx^*),$$

this implies that $d(x^*, Tx^*) = 0$, namely x^* is a fixed point of T.

Suppose y^* is another fixed point of T, then from hypothesis, we have

$$\begin{aligned} d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leq r\{d(x^*, Tx^*) + d(y^*, Ty^*)\} \\ &= 0, \end{aligned}$$

so $d(x^*, y^*) = 0$. This implies that $x^* = y^*$. Hence, T has a unique fixed point $x^* \in X$. Since x_0 is a arbitrary element in X we can deduce $T^n x \to x^*$ for any $x \in X$. □

Theorem 5. *Let (X, d, K) be a strong b-metric space. If every map $T : X \to X$ satisfying*

$$d(Tx, Ty) \leqslant r\big(d(x, Tx) + d(y, Ty)\big)$$

for all $x, y \in X$, $r \in [0, \frac{1}{2})$, has a unique fixed point, then (X, d, K) must be a complete strong b-metric space.

Proof. Assume that X is not complete, then there is a Cauchy sequence $\{x_n\}$ in X, which is not convergent in X. Without loss of generality, we assume that all terms of the sequence $\{x_n\}$ are distinct. Now we consider the following set:

$$A = \{x_n : n \in \mathbb{N}\}.$$

For $x \in X$, we set

$$d(x, A) = \inf_{a\in A} d(x, a).$$

Next we prove $d(x, A) > 0$ for all $x \in X\backslash A$. Indeed, assuming the opposite there exists $\bar{x} \in X\backslash A$ such that $d(\bar{x}, A) = 0$. From definition of $d(\bar{x}, A)$ we have that there exists a sequence $\{x_{n_k}\} \subset \{x_n\}$ satisfying

$$\lim_{k\to\infty} d(\bar{x}, x_{n_k}) = 0.$$

For every $\varepsilon > 0$, since $\{x_n\}$ is Cauchy sequence so we have that there exists n_0 such that $d(x_n, x_m) < \varepsilon/2$ if all $m, n \geqslant n_0$. And since $\lim_{k\to\infty} d(\bar{x}, x_{n_k}) = 0$, there exists k_0 such that $d(\bar{x}, x_{n_k}) < \varepsilon/2K$ if every $k \geqslant k_0$. We choose k_1 satisfying $k_1 \geqslant k_0$ and $n_{k_1} \geqslant n_0$. Then for every $n \geqslant n_0$, we have

$$d(x_n, \bar{x}) \leqslant d(x_n, x_{n_{k_1}}) + Kd(x_{n_{k_1}}, \bar{x}) < \varepsilon.$$

Hence $\lim_{n\to\infty} x_n = \bar{x}$, it is a contraction.

Now let $x \in X$ be a arbitrary element. If $x \in X\backslash A$, then $d(x, A) > 0$ so since $\{x_n\}$ is Cauchy sequence we can find an integer $n_x \in \mathbb{N}$ such that

$$d(x_m, x_{n_x}) \leqslant rd(x, A)$$

for all $m \geqslant n_x$. From definition we have $d(x, A) \leqslant d(x, x_n)$ for all $n \in \mathbb{N}$. This implies that

$$d(x_m, x_{n_x}) \leqslant rd(x, x_n) \tag{4}$$

for all $m \geqslant n_x$ and $n \in \mathbb{N}$.

If $x \in A$ then $x = x_{n_0}$ for some $n_0 \in \mathbb{N}$. So $x \notin A_1 = A\backslash\{x_1, x_2, \ldots, x_{n_0}\}$. Again, we can $n_0' \in \mathbb{N}$, $n_0' > n_0$ such that

$$d(x_m, x_{n_0'}) \leqslant rd(x_{n_0}, x_n)$$

for all $m \geqslant n_0' > n_0$ and $n > n_0$. Specially

$$d(x_m, x_{n_0'}) \leqslant rd(x_{n_0}, x_{n_0'}) \tag{5}$$

for any $m \geqslant n_0' > n_0$.

Now we define $T : X \to X$ by

$$Tx = \begin{cases} x_{n_x} & \text{if } x \in X\backslash A, \\ x_{n_0'} & \text{if } x \in A \text{ and } x = x_{n_0}. \end{cases}$$

Obviously, T has no fixed point.

Let x, y be arbitrary points with $x \neq y$, we show that

$$d(Tx, Ty) \leqslant r(d(x, Tx) + d(x, Ty)). \tag{6}$$

Indeed, we consider three cases as possible:

(i) $x, y \in X\backslash A$, then $Tx = x_{n_x}$ and $Ty = x_{n_y}$. Without loss of generality, we assume that $n_y \geqslant n_x$. Then from (4), we get

$$\begin{aligned} d(Tx, Ty) &= d(x_{n_x}, x_{n_y}) \\ &\leqslant rd(x, x_{n_x}) \\ &= rd(x, Tx). \end{aligned} \tag{7}$$

(ii) $x \in X\backslash A$ and $y \in A$ then $Tx = x_{n_x}$. Set $y = x_{n_0}$ for some n_0, then we obtain $Ty = x_{n_0'}$. If $n_0' \geqslant n_x$ then from (4), we have

$$\begin{aligned} d(Tx, Ty) &= d(x_{n_x}, x_{n_0'}) \\ &\leqslant rd(x, x_{n_x}) \\ &= rd(x, Tx). \end{aligned} \tag{8}$$

If $n_0' < n_x$ then from (5), we have

$$\begin{aligned} d(Tx, Ty) &= d(x_{n_x}, x_{n_0'}) \\ &\leqslant rd(x_{n_0}, x_{n_0'}) \\ &= rd(y, Ty). \end{aligned} \tag{9}$$

(iii) $x, y \in A$, then $x = x_{n_0}, y = y_{m_0}$ for some $n_0 \neq m_0 \in \mathbb{N}$. So $Tx = x_{n_0'}$, $Ty = x_{m_0'}$. Without loss of generality we assume that $m_0' \geqslant n_0'$. Since (5), we have then from (5), we have

$$\begin{aligned} d(Tx, Ty) &= d(x_{n_0'}, x_{m_0'}) \\ &\leqslant rd(x_{n_0}, x_{n_0'}) \\ &= rd(x, Tx). \end{aligned} \tag{10}$$

Combining (7), (8), (9), (10) we obtain (6). Therefore, for all $x, y \in X$ with $x \neq y$, we have

$$d(Tx, Ty) \leqslant r(d(x, Tx) + d(x, Ty)),$$

namely, T is a Kannan map which has no fixed point. This is a contraction. Hence X must be a complete strong b-metric space. □

Remark: We know that, a strong b-metric space is metric space when $K = 1$, so Theorem 4 is a generalization of Theorem 1. Combining two above theorem we have that a strong b-metric space is complete if and only if every Kannan map has a fixed point.

3. Conclusion

In this paper, we prove that a strong b-metric space is complete if and only if every Kannan map has a fixed point. This result is generalization of result of Subrahmanyam in [4].

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