

METHOD OF INFINITE SYSTEM OF EQUATIONS ON NON-UNIFORM GRIDS FOR SOLVING A BOUNDARY PROBLEM FOR ELLIPTIC EQUATION IN A SEMISTRIP

Tran Dinh Hung*

University of Education - TNU

ABSTRACT

For solving boundary value problems in unbounded domains, one usually restricts them to bounded domains and find ways to set appropriate conditions on artificial boundaries or use quasi-uniform grid that maps unbounded domains to bounded ones. Differently from the above methods we approach to problems in unbounded domains by infinite systems of equations. Some initial results of this method are obtained for several 1D problems. Recently, we have developed the method for an elliptic problem in a semistrip. Using the idea of Polozhii in the method of summary representations we transform infinite system of three-point vector equations to infinite systems of three-point scalar equations and show how to obtain an approximate solution with a given accuracy. In this paper we continue to develop the method on non-uniform grids for solving a boundary problem for elliptic equation in a semistrip.

Key word: unbounded domain; elliptic equation; infinite system; method of summary representation; non-uniform grid

INTRODUCTION

A number of mechanical as well as physical problems are posed in infinite (or unbounded) domains. In order to solve these problems, many authors often limit themselves to deal with the problem in a finite domain and make effort to use available efficient methods for finding exact or approximate solution in the restricted domain. But there are some questions which arise: how large size of restricted domain is adequate and how to set conditions on artificial boundary to achieve approximate solution with good accuracy? Mathematicians often try to define appropriate conditions on the boundary. These boundary conditions are called artificial or absorbing boundary conditions (ABCs) ([1], [9]). It is important notice that all the ABCs or TBCs are often constructed for the problems, where the right-hand side function and the initial conditions are assumed to have compact support in space.

Differently from the above method we approach to problems in unbounded domains by infinite system of equations [6]. Some initial results of this method are obtained for a stationary problem of air pollution [2], [3] and

several one-dimensional nonstationary problems [4]. Very recently, in [5] we have successfully developed the approach for an elliptic problem in a semistrip. Using the idea of Polozhii in the method of summary representations we transform infinite system of three-point vector equations to infinite systems of three-point scalar equations and show how to obtain an approximate solution with a given accuracy. But in the mentioned works due to the use of uniform grids (UGs) on the whole unbounded domains the efficiency of our method is limited. In the Conclusion of [5] we highlighted the way to overcome this shortcoming. It is the use of non-uniform grids (NUGs) with monotonically increasing grid sizes.

In this paper we continue to develop the method on non-uniform grids for solving a boundary problem for elliptic equation [10] in a semistrip:

$$\begin{aligned} Lu &= \gamma \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial^2 u}{\partial y^2} + a(x) \frac{\partial u}{\partial x} - b(x)u(x, y) \\ &= f(x, y), \quad x > 0, 0 < y < 1, \\ u(x, 0) &= \varphi_1(x), u(x, 1) = \varphi_2(x), u(0, y) \\ &= \psi(y), u(x, y) \rightarrow 0, \quad x \rightarrow +\infty, \end{aligned} \quad (1)$$

*Tel: 0983 966789, Email: trandinhhungvn@gmail.com

under the usual assumptions that the functions in (1) are continuous and

$$\gamma > 0, \quad |a(x)| \leq r, \quad b(x) \geq 0,$$

$$f(x, y) \rightarrow 0, \varphi_i(x) \rightarrow 0, \quad x \rightarrow +\infty.$$

CONSTRUCTION OF DIFFERENCE SCHEME

In order to solve the problem (1) we introduce on $\bar{\omega} = \{(x, y), x \geq 0, 0 \leq y \leq 1\}$ the non-uniform grid (NUG) in x dimension

$\bar{\omega}_h = \{(x_i, y_j), x_i = x_{i-1} + h_1(i), y_j = jh_2, i = 1, 2, \dots, j = 0, 1, \dots, M\}$ with $x_0 = 0$. Denote the set of interior points

$$\text{by } \omega_h, \quad h_i = \frac{h_1(i) + h_1(i+1)}{2}, \quad i = 0, 1, \dots$$

In sequel we shall use the Samarski technique and notations in [8]. Set

$$L_x u = \gamma \frac{\partial^2 u}{\partial x^2} + a(x) \frac{\partial u}{\partial x} - b(x)u(x, y)$$

and consider the associated perturbed operator

$$\tilde{L}_x u = \gamma \frac{\partial^2 u}{\partial x^2} + a(x) \frac{\partial u}{\partial x} - b(x)u(x, y),$$

$$\text{where } \gamma = \frac{1}{1+R}, \quad R = \frac{1}{2} \frac{h(x) \cdot |a(x)|}{\gamma},$$

$$h(x) = h_1(1) = x_1 - x_0, \quad 0 \leq x < \frac{x_0 + x_1}{2},$$

$$\bar{h}(x) = \bar{h}_i = \frac{x_{i+1} - x_{i-1}}{2},$$

$$\frac{x_{i-1} + x_i}{2} \leq x < \frac{x_i + x_{i+1}}{2}, \quad i = 1, 2, \dots,$$

Now represent the function $a(x)$ as a sum of a nonnegative and a nonpositive terms

$$a = a^+ + a^-, \quad a^+ = \frac{1}{2}(a + |a|) \geq 0,$$

$$a^- = \frac{1}{2}(a - |a|) \leq 0.$$

Denote by v_{ij} the approximation of the values $u(x_i, y_j)$ on the grid $\bar{\omega}_h$,

$$\gamma_i = \gamma(x_i), \quad a_i^+ = a^+(x_i), \quad a_i^- = a^-(x_i),$$

$$b_i = b(x_i), \quad f_{ij} = f(x_i, y_j), \quad (x_i, y_j) \in \bar{\omega}_h.$$

Next, we approximate the operator $\tilde{L}_x u$ by the difference

operator $\hat{L}_x v \equiv \gamma \gamma_{\bar{x}\bar{x}} v + a^+ v_{\hat{x}} + a^- v_{\bar{x}} - b v$, where

$$v_{\hat{x}} = v_{\hat{x}}(x_i, y_j) = \frac{v_{i+1,j} - v_{i,j}}{h_1(i+1)},$$

$$v_{\bar{x}} = v_{\bar{x}}(x_i, y_j) = \frac{v_{i,j} - v_{i-1,j}}{h_1(i)},$$

$$v_{\bar{x}\bar{x}} = v_{\bar{x}\bar{x}}(x_i, y_j) = \frac{1}{h_i} \left(\frac{v_{i+1,j} - v_{i,j}}{h_1(i+1)} - \frac{v_{i,j} - v_{i-1,j}}{h_1(i)} \right)$$

and replace the differential problem (1) by the difference scheme

$$L_h v \equiv \gamma v_{\bar{y}\bar{y}} + \hat{L}_x v = f_{ij}, \quad (x, y) \in \omega_h, \quad (2)$$

$$v_{i,0} = \varphi_1(x_i), \quad v_{i,M} = \varphi_2(x_i), \quad v_{0,j} = \psi(y_j), \quad v_{i,j} \rightarrow 0, \quad i \rightarrow +\infty.$$

Follow [5] and [8] it is easy to see that the difference scheme (2) converges with the accuracy $(O(h_1(i))^2 + h_2^2)$.

SOLUTION METHOD

We write the difference equations in (2) in detailed form

$$\gamma \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{h_2^2} + \gamma_i \gamma \frac{1}{\bar{h}_i} \left(\frac{v_{i+1,j} - v_{i,j}}{h_1(i+1)} - \frac{v_{i,j} - v_{i-1,j}}{h_1(i)} \right)$$

$$+ a_i^+ \frac{v_{i+1,j} - v_{i,j}}{h_1(i+1)} + a_i^- \frac{v_{i,j} - v_{i-1,j}}{h_1(i)} - b_i v_{i,j} = f_{i,j},$$

$$i = 1, 2, \dots; \quad j = 1, 2, \dots, M-1$$

and transform them to the standard five-points difference scheme

$$\left(\frac{\gamma \gamma_i}{\bar{h}_i h_1(i)} - \frac{a_i^-}{h_1(i)} \right) v_{i-1,j} + \frac{\gamma}{h_2^2} (v_{i,j+1} + v_{i,j-1}) + \left(\frac{\gamma \gamma_i}{\bar{h}_i h_1(i+1)} + \frac{a_i^+}{h_1(i+1)} \right) v_{i+1,j} -$$

$$- \left(\frac{\gamma \gamma_i}{\bar{h}_i} \left(\frac{1}{h_1(i+1)} + \frac{1}{h_1(i)} \right) - \frac{a_i^-}{h_1(i)} + \frac{a_i^+}{h_1(i+1)} + b_i + 2 \frac{\gamma}{h_2^2} \right) v_{i,j} = f_{i,j},$$

$$i = 1, 2, \dots; \quad j = 1, 2, \dots, M-1.$$

(3)

$$\text{Put } A_i = \frac{\gamma \gamma_i}{h_i h_1(i)} - \frac{a_i^-}{h_1(i)} \geq 0,$$

$$B_i = \frac{\gamma \gamma_i}{h_i h_1(i+1)} + \frac{a_i^+}{h_1(i+1)} \geq 0 \quad \text{and denote}$$

$$V_0 = \begin{pmatrix} \psi(y_1) \\ \psi(y_2) \\ \dots \\ \psi(y_{M-1}) \end{pmatrix}, \quad V_i = \begin{pmatrix} v_{i,1} \\ v_{i,2} \\ \dots \\ v_{i,M-1} \end{pmatrix}, \quad \bar{F}_i = \begin{pmatrix} f_{i,1} - \frac{\gamma}{h_2^2} v_{i,0} \\ f_{i,2} \\ \dots \\ f_{i,M-2} \\ f_{i,M-1} - \frac{\gamma}{h_2^2} v_{i,M} \end{pmatrix}, \quad i = 1, 2, \dots,$$

Then the equations (3) together with boundary conditions can be written in the form of three-point vector difference equations

$$A_i V_{i-1} + \frac{\gamma}{h_2^2} T V_i + B_i V_{i+1} - (A_i + B_i + b_i + 2 \frac{\gamma}{h_2^2}) V_i = \bar{F}_i \quad i=1,2,\dots \quad (4)$$

where V_0 is defined above, $V_i \rightarrow 0$ as $i \rightarrow +\infty$ and T is the matrix of order $M-1$

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

Next, we shall use the idea of Polozhii in the method of summary representations [7] to transform the infinite system of three-point vector equations (4) to infinite systems of three-point scalar equations. For this purpose let us introduce the notations

$$S = (s_{ij})_1^{M-1}, \quad s_{ij} = \sqrt{\frac{2}{M}} \sin \frac{ij\pi}{M},$$

$$i, j = 1, 2, \dots, M-1,$$

$$\Lambda = [\lambda_1, \lambda_2, \dots, \lambda_{M-1}],$$

$$\lambda_j = 2 \cos \frac{j\pi}{M}, \quad j = 1, 2, \dots, M-1.$$

We have $S^T = S$, $S^2 = E$ and $T = S^{-1} \Lambda S$.

Multiplying both sides of (4) with the matrix S and put $W_i = (w_{i,j}) = S V_i$, $G_i = (g_{i,j}) = S \bar{F}_i$, $i = 0, 1, 2, \dots$, $j = 1, 2, \dots, M-1$, we obtain

$$A_i W_{i-1} + \frac{\gamma}{h_2^2} \Lambda W_i + B_i W_{i+1} - (A_i + B_i + b_i + 2 \frac{\gamma}{h_2^2}) W_i = G_i, \quad i = 1, 2, \dots$$

For every fixed index j we have the system

$$A_i w_{i-1,j} - (A_i + B_i + b_i + \frac{4\gamma}{h_2^2} \sin^2 \frac{j\pi}{2M}) w_{i,j} + B_i w_{i+1,j} = g_{i,j}, \quad i = 1, 2, \dots \quad (5)$$

$$w_{0,j} = \mu_0 = \sum_{l=1}^{M-1} s_{j,l} v_{0,l}, \quad w_{i,j} \rightarrow 0, \quad i \rightarrow \infty.$$

It is obvious that (5) has the form of customary three-point difference equations

$$A_i w_{i-1,j} - C_{i,j} w_{i,j} + B_i w_{i+1,j} = -F_{i,j}, \quad i = 1, 2, \dots \quad (6)$$

$$w_{0,j} = \mu_0, \quad w_{i,j} \rightarrow 0, \quad i \rightarrow \infty,$$

$$F_{ij} = -g_{ij} \text{ and}$$

$$C_{i,j} = A_i + B_i + b_i + \frac{4\gamma}{h_2^2} \sin^2 \frac{j\pi}{2M} > 0.$$

Therefore, the solution of the system (3) is reduced to the solution of $M-1$ systems of customary three-point difference equations (6).

Next, in order to treat the system (6) we shall use the method of infinite system of equations in [6], which was developed by ourselves for solving some one-dimensional problems in [4]. For this purpose we set

$$p_{0,j} = q_{0,j} = 0, \quad r_{0,j} = \mu_0, \quad p_{i,j} = \frac{A_i}{C_{i,j}}, \quad (7)$$

$$q_{i,j} = \frac{B_i}{C_{i,j}}, \quad r_{i,j} = \frac{F_{i,j}}{C_{i,j}}, \quad i = 1, 2, \dots$$

and rewrite the system (7) in the canonical form of infinite system as follows

$$w_{i,j} = p_{i,j} w_{i-1,j} + q_{i,j} w_{i+1,j} + r_{i,j}, \quad i = 0, 1, 2, \dots \quad (8)$$

$$w_{i,j} \rightarrow 0, \quad i \rightarrow \infty.$$

It is easy to see that the conditions of Theorem 2.3 in [4] are satisfied and the solution of the infinite system (8) can be found by the truncation method.

Following the progonka method (or Thomas algorithm) which is a special form of the Gauss elimination [8] for tridiagonal system of equations we shall seek the solution of (8) in the form

$$w_{i,j} = \alpha_{i+1,j} w_{i+1,j} + \beta_{i+1,j}, \quad i = 0, 1, \dots, \quad (9)$$

where coefficients are calculated by the formulas

$$\alpha_{1,j} = 0, \quad \beta_{1,j} = \mu_0, \quad \alpha_{i+1,j} = \frac{q_{i,j}}{1 - p_{i,j} \alpha_{i,j}}, \quad (10)$$

$$\beta_{i+1,j} = \frac{r_{i,j} + p_{i,j} \beta_{i,j}}{1 - p_{i,j} \alpha_{i,j}}, \quad i = 1, 2, \dots$$

From the Theorem 3.2 in [4], we can get the following theorem

Theorem 1. Given an accuracy $\varepsilon > 0$. If starting from a natural number N_j there

holds $\frac{|\beta_{i,j}|}{1 - \alpha_{i,j}} \leq \varepsilon, \forall i \geq N_j + 1$ then for the

deviation of the solution of the truncated system

$$\bar{w}_{i,j} = p_{i,j} \bar{w}_{i-1,j} + q_{i,j} \bar{w}_{i+1,j} + r_{i,j}, i = 0, 1, 2, \dots, N_j, \\ \bar{w}_{i,j} = 0, i \geq N_j + 1$$

compared with the solution w_{ij} of the infinite system (8) there holds the following estimate

$$\sup_i |w_{i,j} - \bar{w}_{i,j}| \leq \varepsilon.$$

Now denote $\bar{V}_i = (\bar{v}_{i,j})_{j=1}^{M-1}, \bar{W}_i = (\bar{w}_{i,j})_{j=1}^{M-1}, i = 0, 1, 2, \dots$ and set $\bar{V}_i = S \bar{W}_i$.

Theorem 2. We have the estimation: $\sup_{i,j} |v_{i,j} - \bar{v}_{i,j}| \leq \sqrt{M-1} \varepsilon$.

The prove of Theorem 2 is similar as the Theorem 4 in [5].

NUMERICAL EXAMPLES

The experiments are performed on NUGs with increased stepsizes $h_1(i) = 1, h_1(i-1), i = 1, 2, \dots, h_2$ is the stepsizes in y dimension. $N = \max\{N_1, N_2, \dots, N_{M-1}\}$ is the size of the system that is automatically truncated with the given accuracy ε , error = $\max_{i,j} |u(x_i, y_j) - \bar{v}_{i,j}|$ is the error of the obtained approximate solution compared with the exact solution.

Example 1. We take

$$\gamma = 1, a(x) = -1, b(x) = 1 + \frac{1}{(x^2 + 1)^2}, \\ u = \frac{\exp(-y/\sqrt{\gamma}) + \exp((y-1)/\sqrt{\gamma}) + x}{x^2 + 1}.$$

The results of convergence are given in the Table 1. Remark that in [10] the equation (1) was considered in the whole strip $(-\infty < x < +\infty, 0 \leq y \leq 1)$. It was discretized and the obtained three-point system of vector equations was truncated. The numerical

experiment for the above example gave the error of the truncated system at large number of equations in comparison with the infinite system but there was no information of its deviation from the exact solution.

Table 1. The convergence of the method in Example 1

$h_1(0)$	h_2	ε	N	error
1	0,1	0,1	15	0,0089
0,01	0,04	0,01	89	0,0067
0,01	0,01	0,01	96	$6,14 \cdot 10^{-4}$

Example 2. In this example, we do not know the exact solution of the problem. Now we take

$$\gamma = 0.1, a(x) = -\frac{1}{1+x^2}, b(x) = 0, f(x, y) = 0, \\ \varphi_1(x) = e^{-x/5}, \varphi_2(x) = e^{-x/10}, \psi(y) = 1 + \frac{\sin(\pi y)}{2}.$$

We have the results of convergence given in the Table 2.

Table 2. The convergence of the method in Example 2

$h_1(0)$	h_2	ε	N
1	0,1	0,1	14
0,1	0,1	0,01	36
0,1	0,01	0,01	39

CONCLUSION

In this paper, we developed the numerical method in [5] by using the non-uniform grids with monotonically increasing grid sizes. Some numerical examples are shown to illustrate the effectiveness of the method.

The development of the method for solving other two-dimensional and three-dimensional problems is the direction of our research in the future.

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TÓM TẮT

PHƯƠNG PHÁP HỆ VÔ HẠN TRÊN LƯỚI KHÔNG ĐỀU GIẢI MỘT BÀI TOÁN BIÊN CHO PHƯƠNG TRÌNH ELLIPTIC TRONG NỬA DẢI

Trần Đình Hùng*

Trường Đại học Sư phạm - ĐH Thái Nguyên

Để giải số các bài toán biên trong miền vô hạn, người ta thường giới hạn bài toán trong một miền hữu hạn và tìm cách thiết lập các điều kiện biên xấp xỉ trên biên nhân tạo hoặc sử dụng lưới tính toán tựa đều ánh xạ miền không giới nội vào miền giới nội. Khác với các cách làm trên, chúng tôi tiếp cận tới bài toán trong miền không giới nội bởi hệ vô hạn các phương trình đại số tuyến tính. Một số kết quả ban đầu đối với các bài toán một chiều đã được công bố. Gần đây, chúng tôi đã đề xuất phương pháp giải một bài toán elliptic trong nửa dải. Sử dụng ý tưởng của Polozhii trong phương pháp biểu diễn tổng, chúng tôi đã đưa được hệ phương trình véc tơ ba điểm về các hệ phương trình sai phân vô hướng ba điểm và thu nhận được nghiệm gần đúng của bài toán với sai số cho trước. Trong bài báo này chúng tôi tiếp tục phát triển phương pháp trên lưới không đều giải một bài toán biên cho phương trình elliptic trong nửa dải.

Từ khóa: miền vô hạn, phương trình elliptic, hệ vô hạn, phương pháp biểu diễn tổng, lưới không đều

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* Tel: 0983 966789, Email: trandinhhungvn@gmail.com