On Uniqueness of $p-$adic Meromorphic Function Concerning Differential Polynomials

Ha Tran Phuong
Department of Mathematics, Thai Nguyen University of Education
Email: hatranphuong@yahoo.com
Nguyen Van Thin
Department of Mathematics, Thai Nguyen University of Education
nguyenvanthintn@gmail.com

ABSTRACT

In this paper, we study the uniqueness problem on difference polynomials and its differential of p-adic meromorphic function sharing a common value.
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1. Introduction

Recently the problem of studying the value distribution of differential and difference polynomials is developed by many authors. For example: Yang and Hua ([5]), Halburd and Korhonen ([3]), Laine and Yang ([4]), Khoai and An ([1]), ... In this paper we prove some results about uniqueness problem on differential of p-adic meromorphic function sharing a common value. First, we introduce the standard notation of $p-$adic Nevanlinna theory.

Let $\mathcal{A}(\mathbb{C}_p)$ be the ring of entire function on $\mathbb{C}_p$. Then each $f\in\mathcal{A}(\mathbb{C}_p)$ can be given a power series $f(z)=\sum_{n=0}^{\infty}a_nz^n$ Then the *maximum* term of f is defined to be $|f|_r=\max_{n\geqslant 0}\{|a_n|r^n\}$.

Let $n(r,\dfrac{1}{f})$ be a number of zeros (counting multiplicities) of f in the disk $\mathbb{C}_p[0,r]=\{z\in\mathbb{C}_p:|z|\leqslant r\}$. Fix a real number $\rho_0>0$. For $r>\rho_0$, we denote by $N(r,\dfrac{1}{f})$ the counting function of zeros f and by $\overline{N}(r,\dfrac{1}{f})$ the counting function of zeros f ignoring multiplicities. Let a be in $\mathbb{C}_p$, we denote the number distinct $a-$point of f in $\mathbb{C}_p[0,r]$ by $\overline{n}(r,\dfrac{1}{f-a})$. We denote by $\overline{N}_{(m}(r,a;f)$, (or $\overline{N}_{(m}(r,\dfrac{1}{f-a})$) the reduced counting function of a-point of f whose multiplicities are not less than m. Namely

$$\overline{N}_{(m}(r,\frac{1}{f-a})=\int_{\rho_0}^{r}\frac{\overline{n}_{(m}(t,\frac{1}{f-a})}{t}dt.$$

Let f be a non-constant p-adic meromorphic function. This means $f=\dfrac{f_1}{f_2}$, $f_1,f_2\in\mathcal{A}(\mathbb{C}_p)$. We denote by $\overline{N}_{(m}(r,f)$ the reduced counting function of poles of f whose multiplicities are at least m. Namely

$$\overline{N}_{(m}(r,f)=\int_{\rho_0}^{r}\frac{\overline{n}_{(m}(t,f)}{t}dt,$$

where $\overline{n}_{(m}(r,f)=\overline{n}_{(m}(r,\frac{1}{f_2})$. Let k be a positive integer and $a\in\mathbb{C}_p$, we set

$$N_k(r,\frac{1}{f-a})=\overline{N}(r,\frac{1}{f-a})+\overline{N}_{(2}(r,\frac{1}{f-a})+\cdots+\overline{N}_{(k}(r,\frac{1}{f-a}),$$
$$N_2(r,f)=\overline{N}(r,f)+\overline{N}_{(2}(r,f).$$

Our results are stated as follows:

Main Theorem. *Let f, g be p-adic transcendental entire functions and $k\geqslant 1$, $t\geqslant 1, n\geqslant 2k+4+t$ are integers. If $(f^n(z)f(z+b_1)\dots f(z+b_t))^{(k)}$ and $(g^n(z)g(z+b_1)\dots g(z+b_t))^{(k)}$ share $1-CM$, where and $b_1,\dots,b_t$ are nonzero distinct constants. Then $f\equiv hg$, where $h^{n+t}=1$.*

2. Some Lemmas

In order to prove theorems, we need the following lemmas.

Lemma 2.1. ([1]) *Let f be a non-constant p-adic meromorphic function. Then*

$$m(r,\frac{f(z+c)}{f(z)})=O(1) \text{ and } T(r,f(z+c))=T(r,f)+O(1).$$

Let f and g are two non-constant p-adic meromorphic functions, we set

$$H=\frac{f''}{f'}-2\frac{f'}{f-1}-\frac{g''}{g'}+2\frac{g'}{g-1}.$$

Lemma 2.2.([1]) *If $H\not\equiv 0$ and f and g share 1-CM then*

$$T(r,f)\leqslant N_2(r,f)+N_2(r,\frac{1}{f})+N_2(r,g)+N_2(r,\frac{1}{g})-\log r+O(1)$$
$$T(r,g)\leqslant N_2(r,f)+N_2(r,\frac{1}{f})+N_2(r,g)+N_2(r,\frac{1}{g})-\log r+O(1).$$

By using the property of the $p-$adic Nevanlinna functions, we easy prove the following lemmas.

Lemma 2.3. *Let f be a non-constant $p-$adic meromorphic function and $f^{(k)}\not\equiv 0$. Then we have*

$$\overline{N}(r,\frac{1}{f^{(k)}})\leqslant k\overline{N}(r,f)+N_{k+1}(r,\frac{1}{f})+O(1); \tag{2.1}$$
$$\overline{N}(r,\frac{1}{f^{(k)}})\leqslant N_{k+1}(r,\frac{1}{f})+T(r,f^{(k)})-T(r,f)+O(1); \tag{2.2}$$
$$N_2(r,\frac{1}{f^{(k)}})\leqslant k\overline{N}(r,f)+N_{k+2}(r,\frac{1}{f})+O(1); \tag{2.3}$$
$$N_2(r,\frac{1}{f^{(k)}})\leqslant N_{k+2}(r,\frac{1}{f})+T(r,f^{(k)})-T(r,f)+O(1). \tag{2.4}$$

3. Proof of Main Theorem

First we prove statement 1. Set

$$F(z) = f^n(z)f(z+b_1)\dots f(z+b_t), \quad G(z) = g^n(z)g(z+b_1)\dots g(z+b_t).$$

By hypothesis, we have $F^{(k)}$ and $G^{(k)}$ share $1 - CM$. Set

$$H = \frac{F^{(k+2)}}{F^{(k+1)}} - 2\frac{F^{(k+1)}}{F^{(k)}-1} - \frac{G^{(k+2)}}{G^{(k+1)}} + 2\frac{G^{(k+1)}}{G^{(k)}-1}.$$

We first prove $H \equiv 0$. Indeed if $H \not\equiv 0$, by Lemma 2.2, we get

$$\begin{aligned}(3.1)\qquad T(r,F^{(k)})+T(r,G^{(k)}) \leqslant{}& 2\big(N_2(r,F^{(k)}) + N_2(r,\frac{1}{F^{(k)}})\big)\\ &+ 2\big(N_2(r,G^{(k)}) + N_2(r,\frac{1}{G^{(k)}})\big) - 2\log r + O(1).\end{aligned}$$

By using Lemma 2.3, (3.1) becomes

$$\begin{aligned}(3.2)\qquad T(r,F)+T(r,G) \leqslant{}& 2N_2(r,F^{(k)}) + 2N_2(r,G^{(k)}) + k\overline{N}(r,F) + k\overline{N}(r,G)\\ &+ 2N_{k+2}(r,\frac{1}{F}) + 2N_{k+2}(r,\frac{1}{G}) - 2\log r + O(1).\end{aligned}$$

By hypothesis that $f(z), g(z)$ are entire functions, we have from (3.2)

$$(3.3)\qquad T(r,F)+T(r,G) \leqslant 2N_{k+2}(r,\frac{1}{F}) + 2N_{k+2}(r,\frac{1}{G}) - 2\log r + O(1).$$

By the argument as Lemma 2.1, we have $(r, \frac{f(z)}{f(z+c)}) = O(1), c \neq 0$, so from Lemma 2.1, we have

$$\begin{aligned}m(r,\frac{f(z+c)}{f(z+b_i)}) &= m(r,\frac{f(z+c)}{f(z)}.\frac{f(z)}{f(z+b_i)})\\ &\leqslant m(r,\frac{f(z+c)}{f(z)}) + m(r,\frac{f(z)}{f(z+b_i)}) = O(1)\end{aligned}$$

for any $i \in \{1,\dots,n\}$. So

$$\begin{aligned}(n+t)T(r,f) &= (n+t)T(r,f(z+c)) + O(1) = T(r,f^{n+t}(z+c)) + O(1)\\ &= m(r,f^{n+t}(z+c)) + O(1)\\ &\leqslant m(r,f^n(z+c)f(z+b_1)\dots f(z+b_t))\\ &\quad + m\left(r,\frac{f(z+c)}{f(z+b_1)}\right) + \dots + m\left(r,\frac{f(z+c)}{f(z+b_t)}\right) + O(1)\\ &= m(r,f^n(z+c)f(z+b_1)\dots f(z+b_t)) + O(1)\\ &\leqslant m(r,f^n(z)f(z+b_1)\dots f(z+b_t)) + m\left(r,(\frac{f(z+c)}{f(z)})^n\right) + O(1)\\ &= m(r,f^n(z)f(z+b_1)\dots f(z+b_t)) + O(1).\end{aligned}$$

It is easy to see that

$$m(r,f^n(z)f(z+b_1)\dots f(z+b_t)) \leqslant (n+t)T(r,f) + O(1).$$

Hence

$$(3.4)\qquad T(r,F)=T(r,f^n(z)f(z+b_1)\dots f(z+b_t))=(n+t)T(r,f)+O(1).$$

Similarity we have

$$(3.5)\qquad T(r,G)=T(r,g^n(z)g(z+b_1)\dots g(z+b_t))=(n+t)T(r,g)+O(1).$$

By definition of F and G, we have

$$(3.6)\qquad \begin{aligned} N_{k+2}(r,\frac{1}{F}) &= N_{k+2}(r,\frac{1}{f^n(z)f(z+b_1)\dots f(z+b_t)}) \\ &\leqslant (k+2)\overline{N}(r,\frac{1}{f})+\sum_{j=1}^{t}N(r,\frac{1}{f(z+b_j)}). \end{aligned}$$

$$(3.7)\qquad \begin{aligned} N_{k+2}(r,\frac{1}{G}) &= N_{k+2}(r,\frac{1}{g^n(z)g(z+b_1)\dots g(z+b_t)}) \\ &\leqslant (k+2)\overline{N}(r,\frac{1}{g})+\sum_{j=1}^{t}N(r,\frac{1}{g(z+b_j)}). \end{aligned}$$

Combining (3.3), (3.4), (3.5), (3.6) and (3.7) we obtain

$$(n-(2k+4+t))(T(r,f)+T(r,g))+2\log r\leqslant O(1).$$

This is contradiction with $n\geqslant 2k+4+t$. Hence $H\equiv 0$, so

$$(3.8)\qquad \frac{1}{F^{(k)}-1}=\frac{a}{G^{(k)}-1}+b,$$

where $a,b\in\mathbb{C}_p$ are constants, $a\neq 0$. From (3.8), we get

$$(3.9)\qquad F^{(k)}=\frac{(b+1)G^{(k)}+a-b-1}{bG^{(k)}+a-b};\quad G^{(k)}=\frac{(b-a)F^{(k)}+a-b-1}{bF^{(k)}-(b+1)}.$$

We consider the following cases

Case 1. $b\neq 0$. If $a=b=-1$, we get $F^{(k)}.G^{(k)}=1$. This implies that

$$(f^n(z)f(z+b_1)\dots f(z+b_t))^{(k)}.(g^n(z)g(z+b_1)\dots g(z+b_t))^{(k)}=1.$$

In this case f and g have no zeros. Indeed if z_0 be a zero of f with multiple $p\geqslant 1$, then z_0 is the zero of $(f^n(z)f(z+b_1)\dots f(z+b_t))^{(k)}$ with multiple at least $np-k>0$. Hence z_0 is the pole of $(g^n(z)g(z+b_1)\dots g(z+b_t))^{(k)}$, so z_0 is the pole of one of g, $g(z+b_1)$,..., $g(z+b_t)$. Since g is entire function, $g(z+b_1)$,..., $g(z+b_t)$ and g have no poles. Hence f has no zeros, so f is constant. This is contradiction.

If $a=b\neq -1$, from (3.9), we have

$$(3.10)\qquad \frac{1}{F^{(k)}}=\frac{bG^{(k)}}{(b+1)G^{(k)}-1},\quad G^{(k)}=\frac{-1}{bF^{(k)}-(b+1)}.$$

So

$$\overline{N}(r,\frac{1}{F^{(k)}})=\overline{N}\Big(r,\frac{1}{G^{(k)}-\frac{1}{b+1}}\Big)\leqslant k\overline{N}(r,F)+N_{k+1}(r,\frac{1}{F})+O(1).$$

$$\overline{N}(r,G^{(k)})=\overline{N}\Big(r,\frac{1}{F^{(k)}-\frac{b+1}{b}}\Big)=0.$$

By the second main theorem for entire function $F^{(k)}$, we have

$$(3.11)\qquad T(r, F^{(k)}) \leqslant \overline{N}(r, \frac{1}{F^{(k)}}) + \overline{N}\left(r, \frac{1}{F^{(k)} - \frac{b+1}{b}}\right) - \log r + O(1).$$

By Lemma 2.3, (3.11) becomes

$$(3.12)\qquad \begin{aligned} T(r, F) &\leqslant N_{k+1}(r, \frac{1}{F}) - \log r + O(1) \\ &\leqslant (k + 1 + t)T(r, f) - \log r + O(1). \end{aligned}$$

Combining (3.4) and (3.12), we have $(n-k-1)T(r, f)+\log r \leqslant O(1)$. This is a contradiction with $n \geq 2k + 4 + t$.

If $a \neq b$, $b = -1$. From (3.9), we have

$$F^{(k)} = \frac{a}{-G^{(k)} + a + 1}, \quad \frac{1}{G^{(k)}} = \frac{-F^{(k)}}{-(a + 1)F^{(k)} + a}.$$

If $a \neq b$, $b \neq -1$. From (3.9), we have

$$F^{(k)} - (1 + \frac{1}{b}) = \frac{-a}{b^2(G^{(k)} + \dfrac{a - b}{b})}, \quad G^{(k)} = \frac{(b - a)F^{(k)} + a - b - 1}{bF^{(k)} - (b + 1)}.$$

By the analog method as above, we get a contradiction.

Case 2. $b = 0$, since (3.8) we have

$$(3.13)\qquad F^{(k)} = \frac{1}{a}G^{(k)} + 1 - \frac{1}{a}.$$

This implies that

$$(3.14)\qquad F = \frac{1}{a}G + Q(z),$$

where $Q(z)$ is the polynomial with the order is at most k. It is easy to get

$$T(r, g) = T(r, f) + S(r, f).$$

If $Q(z) \not\equiv 0$. By the second main theorem for p-adic small function, we obtain

$$T(r, F) \leqslant \overline{N}(r, F) + \overline{N}(r, \frac{1}{F}) + \overline{N}(r, \frac{1}{F - Q}) + S(r, f).$$

It implies that $(n - t - 2)T(r, f) \leqslant S(r, f)$. This is a contradiction with $n \geq 2k + 4 + t$. Then $Q \equiv 0$, thus $F = \frac{1}{a}G$. So $F^{(k)} = \frac{1}{a}G^{(k)}$.

From (3.13), we get $1 - \frac{1}{a} = 0$, thus $a = 1$, implies that $F \equiv G$. So we have

$$(3.15)\qquad f^n(z)f(z + b_1)\dots f(z + b_t) = g^n(z)g(z + b_1)\dots g(z + b_t).$$

Let $h = \frac{f}{g}$. If h is not constant, then $h^n(z) = \frac{1}{h(z + b_1)\dots h(z + b_t)}$. By Lemma 2.1, we get $nT(r, h) = tT(r, h) + O(1)$. This is contradiction with $n \geq 2k + 4 + t$. Then h is the constant. This implies that $f = hg$, where $h^{n+t} = 1$. Main Theorem is proved.

Vấn đề duy nhất cho các hàm phân hình $p-$adic dưới điều kiện đa thức đạo hàm

Hà Trần Phương

Giảng viên khoa Toán, Trường Đại học Sư phạm - ĐH Thái Nguyên

Email: hatranphuong@yahoo.com

Nguyễn Văn Thìn

Giảng viên khoa Toán, Trường Đại học Sư phạm - ĐH Thái Nguyên.

nguyenvanthintn@gmail.com

TÓM TẮT

Thời gian gần đây có khá nhiều tác giả đã công bố kết quả về vấn đề duy nhất cho hàm phân hình p-adic. Trong bài báo này chúng tôi chứng minh một kết quả về vấn đề duy nhất cho hàm phân hình p-adic dưới điều kiện đa thức đạo hàm bậc cao của nó.

Key words: Vấn đề duy nhất, Đa thức đạo hàm.

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