LOCAL CAUCHY PROBLEM FOR YANG-MILLS HEAT FLOWS

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ABSTRACT

In this paper, we are interested in the following nonlinear heat equation
$\frac{\partial u}{\partial t} = \partial_r^2 u + \frac{d+1}{r}\partial_r u - 3(d-2)u^2 - (d-2)r^2u^3, (x,t) \in \mathbb{R}_+^2,$
studied by Grotowski in 2001 in studying the special solutions for s Yang-Mills heat flows. We aim to study the Cauchy problem for the equation. We reply on the regularity of the semi-group $\{e^{t\Delta}, t > 0\}$. Firstly, the model is not compatible with the semi-group, we need to transfer to another one and the main difficulty is to handle the unbounded nonlinearity r^2u^3. Finally, we proved the Cauchy problem for the model that for any initial data $u_0 \in W^{1,\infty}_{1+r^2}(\mathbb{R}_+)$, there exists $T(u_0) > 0$ such that the above equation has a unique solution $u(t) \in W^{1,\infty}_{1+r^2}(\mathbb{R}_+)$ for all $t \in [0, T(u_0)]$.

BÀI TOÁN CAUCHY ĐỊA PHƯƠNG CHO CÁC DÒNG NHIỆT YANG-MILLS

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TỪ KHÓA

Bài toán Cauchy địa phương
Vấn đề tồn tại duy nhất địa phương
Dòng nhiệt Yang-Mills
Liên thông Yang-Mills
Dòng nhiệt hình học

TÓM TẮT

Trong bài nghiên cứu này, chúng tôi xem xét phương trình

$$\frac{\partial u}{\partial t} = \partial_r^2 u + \frac{d+1}{r}\partial_r u - 3(d-2)u^2 - (d-2)r^2u^3,$$

được Grotowski đưa ra vào năm 2001 để nghiên cứu các nghiệm đặc biệt của các dòng nhiệt Yang-Mills trên các đa tạp Riemann. Chúng tôi nghiên cứu bài toán Cauchy cho phương trình trên trong không gian $W^{1,\infty}_{1+r^2}(\mathbb{R}_+)$. Phương pháp nghiên cứu dựa trên tính chính quy của nửa nhóm giải tích $\{e^{t\Delta}, t > 0\}$. Bài toán trên không tương thích với nửa nhóm giải tích đã nêu. Bằng cách chuyển đổi bài toán trên thành một bài toán thứ 2 tương thích với nửa nhóm giải tích và tìm ra một hàm trọng thích hợp để điều khiển đại lượng phi tuyến không bị chặn r^2u^3 như là khó khăn chính trong bài toán. Cuối cùng, chúng tôi đã giải quyết hoàn toàn bài toán Cauchy trong không gian $W^{1,\infty}_{1+r^2}(\mathbb{R}_+)$, tức là với mọi giá trị ban đầu $u_0 \in W^{1,\infty}_{1+r^2}(\mathbb{R}_+)$, tồn tại $T(u_0) > 0$ sao cho bài toán tồn tại duy nhất nghiệm $u(t) \in W^{1,\infty}_{1+r^2}(\mathbb{R}_+)$, $t \in [0, T(u_0)]$.

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1. Introduction

In this work, we focus on the following nonlinear heat equation

$$\begin{cases}\frac{\partial u}{\partial t} = \partial_r^2 u + \frac{d+1}{r}\partial_r u - 3(d-2)u^2 - (d-2)r^2u^3 \\ u(0) = u_0 \in L^\infty(\mathbb{R}_+)\end{cases} \quad (1)$$

where $u: (r,t) \in \mathbb{R}_+^2 \to \mathbb{R}$ and $d \in \mathbb{N}^*$. The equation (1) was arisen from the study of Yang-Mills connections which is an important problem in Yang-Mills theory. The theory appears in the study of the weak nuclear force, governing some particles' atomic decay, which is a non-commutative version of Maxwell's electromagnetism, see [1], and [2]. The study of Yang-Mills connections usually is considered in abstract spaces such as Riemann manifolds which get a lot of inconveniences. For that reason, the author in [3] reduced the problem to the simpler version as in (1). We kindly refer the readers to references [4]-[6] for an entirely mathematical introduction and a deeply mathematical study on the solutions of equation (1).

The main goal of this paper is to study the local Cauchy problem to (1). We firstly mention the works of [7] and [8] in which the authors proved the problem in Sobolev spaces H^s. Recently, we have [5] which the authors constructed blowup solutions to (1) and solved the local problem in C_0^∞ defined as the set of smooth functions with compact supports. However, it remains open if the local Cauchy problem can be solved $L^\infty(\mathbb{R}_+)$ or subspaces of $L^\infty(\mathbb{R}_+)$ such as $W^{1,\infty}_{1+r^2}(\mathbb{R}_+)$ and $L^\infty_{1+r^2}(\mathbb{R}_+)$. This work aims to prove the problem in Sobolev space $W^{1,\infty}_{1+r^2}(\mathbb{R}_+)$, a subspace of $L^\infty(\mathbb{R}_+)$ which is strictly more significant than $C_0^\infty(\mathbb{R}_+)$ handled in [5].

2. Mathematical approach

In this section, we aim to analyze the local Cauchy problem for equation (1) that directs us to rigorous proof of the results in our paper. For more convenience, we firstly mention below some necessary notations which we use in the proof of the paper. Let Ω be a Lebesgue measurable set in $\mathbb{R}^n$, we denote $L^\infty(\Omega)$ as the set of all Lebesgue measurable functions on Ω such that

$$\inf\{B, \text{ such that } \mu(\{x: |f(r)| > B\}) = 0\} < +\infty,$$

which is a Banach space with the following norm

$$\| f \|_{L^\infty(\Omega)} = \inf\{B \text{ sao cho } \mu(x \in \Omega: |f(x)| > B) = 0\}.$$

Similarly, we also define $L^\infty_{1+|x|^2}(\Omega)$ by

$$L^\infty_{1+|x|^2}(\Omega) = \left\{f \in L^\infty(\Omega) \text{ sao cho } \|(1+r^2)f\|_{L^\infty(\Omega)} < +\infty\right\},$$

which is also a Banach space with the norm

$$\| f \|_{L^\infty_{1+|x|^2}(\Omega)} = \|(1+|x|^2)f\|_{L^\infty(\Omega)}.$$

In addition to that, we also denote the Banach space $W^{1,\infty}(\Omega)$ by

$$W^{1,\infty}(\Omega) = \{f \in L^\infty(\Omega) \text{ such that } f, |\nabla f| \in L^\infty(\Omega)\}.$$

and weighted Sobolev $W^{1,\infty}_{1+|x|^2}(\Omega)$ defined by

$$W^{1,\infty}_{1+|x|^2}(\Omega) = \{f \in L^\infty(\Omega) \text{ such that } (1+|x|^2)f, (1+|x|^2)|\nabla f| \in L^\infty(\Omega)\}.$$

We call f a radially symmetric function on Ω if and only if for all orthogonal matrices A and $x \in \Omega$, we have $Ax \in \Omega$ and $f(Ax) = f(x)$. We also define $L^\infty_{\text{rad}}(\Omega) = \{z \in L^\infty(\Omega)$ and z radially symmetric $\}$, which is a Banach space with $\|\cdot\|_{L^\infty(\Omega)}$ norm. In particular, we also define the abstract space of Banach-valued functions $\mathcal{X}^T$ for some $T > 0$ by

$$\mathcal{X}^T = L^\infty((0,T], W^{1,\infty}(\Omega) \cap L^\infty_{\text{rad}}(\Omega)),$$

which is also a Banach space with the norm

$$\| z \|_{\mathcal{X}^T} = \sup_{t\in(0,T]} \| z(t) \|_{L^\infty(\mathbb{R}^{d+2})}.$$

Next, we define Δ by

$$\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \cdots + \partial_{x_{d+2}}^2$$

so-called Laplacian operator in $\mathbb{R}^{d+1}$. In particular, taking $\Omega = \mathbb{R}^{d+1}$, we recall the notation of the semi-group $\{e^{t\Delta}\}_{t>0}$ with

$$e^{t\Delta}: L^\infty(\mathbb{R}^{d+2}) \to L^\infty(\mathbb{R}^{d+2}), t > 0$$

and

$$e^{t\Delta} f(x) = \frac{1}{(4\pi t)^{\frac{d+2}{2}}} \int_{\mathbb{R}^{d+2}} e^{-\frac{|x-y|^2}{4t}} f(y) dy. \tag{2}$$

We would like to mention Proposition 48.4 given [9], the following fundamental estimate

$$\|e^{t\Delta} f\|_{L^\infty(\mathbb{R}^{d+2})} \le \| f \|_{L^\infty(\mathbb{R}^{d+2})}. \tag{3}$$

Let us analyze the problem in deriving a good idea to handle our issue. Regarding equation (1), we see that it is so hard to give a direct proof to equation (1) due to the lack of the following linear operator

$$\partial_r^2 u + \frac{d+1}{r} \partial_r u.$$

To overcome this challenge, we used the idea investigated by [10] in which the authors successfully handled the local problem to the harmonic heat flows. Adopting the idea, let us define f be a function defined in $\mathbb{R}_+$, then, we denote $\tilde{f}$ as f's extension to $\mathbb{R}^{d+2}$ defined by

$$\tilde{f}(x) = \begin{cases} f(|x|) \text{ if } x \neq 0, \\ f(0) \text{ if } x = 0. \end{cases} \tag{4}$$

We can see that the extension is always a radially symmetric function. In particular, once $f \in C^2(\mathbb{R}_+)$, then, we have $\tilde{f} \in C^2(\mathbb{R}^{d+2})$ and the following identity holds

$$\Delta \tilde{f} \equiv \partial_r^2 f + \frac{d+1}{r} \partial_r f.$$

We suppose in addition that u is a $C^2(\mathbb{R}_+)$-solution to (1) (so-called by the classical solution), then, the extension $\tilde{u}$ belongs to $C^2(\mathbb{R}^{d+2})$ and $\tilde{u}$ satisfies the following equation

$$\begin{cases} \partial_t \tilde{u} = \Delta \tilde{u} - 3(d-2)\tilde{u}^2 - (d-2)|x|^2 \tilde{u}^3, (x,t) \in \mathbb{R}^{d+2} \times (0,T), \\ \tilde{u}(0) = \tilde{u}_0 \in L^\infty(\mathbb{R}^{d+2}). \end{cases}$$

Note that the solution $\tilde{u}$ remains radially symmetric. Let us introduce

$$z(x,t) = (1 + |z|^2)\tilde{u}(x,t). \tag{5}$$

Thanks to the equation of u in the above, we write

$$\partial_t z = \Delta z + \left[\frac{8|x|^2}{(1+|x|^2)^2} - \frac{2(d+2)}{1+|x|^2} \right] z - \frac{4x \cdot \nabla z}{1+|x|^2} - \frac{3(d-2)}{(1+|x|^2)} z^2 - \frac{(d-2)|x|^2}{(1+|x|^2)^2} z^3.$$

Let us define $\mathcal{F}(z)$ by

$$\mathcal{F}(z) = \left[\frac{8|x|^2}{(1+|x|^2)^2} - \frac{2(d+2)}{1+|x|^2} \right] z - \frac{4x \cdot \nabla z}{1+|x|^2} - \frac{3(d-2)}{(1+|x|^2)} z^2 - \frac{(d-2)|x|^2}{(1+|x|^2)^2} z^3, \tag{6}$$

and it reduces to the following problem

$$\begin{cases} \partial_t z = \Delta z + \mathcal{F}(z), (x,t) \in \mathbb{R}^{d+2} \times (0,T), \\ z(0) = z_0 \in L^\infty(\mathbb{R}^{d+2}), \end{cases} \tag{7}$$

Finally, our problem will follow once the Cauchy problem of (7) is solven.

3. Main results

In this paper, we aim to give the complete proof to the local Cauchy problem to (1). As we analyze in the previous section, it firstly need to prove the local ono for (7):

Proposition 2. 1: *Let* $z_0 \in W^{1,\infty}(\mathbb{R}^{d+2}) \cap L^{\infty}_{rad}(\mathbb{R}^{d+2})$, *then, there exists* $T(z_0) > 0$ *such that problem (8) has a unique solution on* $[0,T]$ *and* $z(t) \in W^{1,\infty}(\mathbb{R}^{d+2}) \cap L^{\infty}_{rad}(\mathbb{R}^{d+2})$. *In particular, the following estimate holds*

$$\| z(t) \|_{W^{1,\infty}(\mathbb{R}^{d+2})} \leq \|z_0\|_{W^{1,\infty}(\mathbb{R}^{d+2})} + 1, \forall t \in [0,T].$$

Proof: The result follows from the Banach fixed point theorem (the unique fixed point of a constructive mapping). Now, we consider arbitrary initial data $z_0 \in W^{1,\infty}(\mathbb{R}^{d+2}) \cap L^{\infty}_{\text{rad}}(\mathbb{R}^{d+2})$ and we introduce a subset in $\mathcal{X}^T$

$$\mathbb{B}_0 = \{z \in \mathcal{X}^T \text{ such that } \|z - e^{t\Delta}(z_0)\|_{\mathcal{X}^T} \leq 1\}.$$

In addition, we also define the mapping $\mathbb{T}$ on $\mathcal{X}^T$ as follows

$$\mathbb{T}(z) = e^{t\Delta} z_0 + \int_0^t e^{(t-s)\Delta\mathcal{F}}(z)(s)ds, z \in \mathcal{X}^T, \tag{8}$$

where $\mathcal{F}(z)$ given as in (7). In the below, we aim to prove that once T small enough, the following properties are valid

(H1): $\mathbb{T}$ maps from $\mathbb{B}_0$ into itself.

(**H2**): $\mathbb{T}$ is a contraction mapping on $\mathbb{B}_0$ i.e. there exists $\lambda \in (0,1)$ such that

$$\|\mathbb{T}(z_1) - \mathbb{T}(z_2)\|_{\mathcal{X}^T} \leq \lambda \|z_1 - z_2\|_{\mathcal{X}^T}, \forall z_1, z_2 \in \mathbb{B}_0.$$

- *Proof of (H1):* Choose $z \in \mathbb{B}_0$ arbitrarily, since $\| z \| \mathcal{X}^T \leq 1$, we derive from (6) the following estimate

$$|\mathcal{F}(z)| \leq C(|z| + |z^2| + |z|^3 + |\nabla z|) \leq C_1(|z| + |\nabla z|) \leq C_1 \| z \|_{\mathcal{X}^T},$$

which yields

$$\| \mathcal{F}(s) \|_{L^\infty(\mathbb{R}^{d+2})} \leq C_1 \| z \|_{\mathcal{X}^T}, \forall s \in (0,T].$$

Besides that, we apply (3) and the above estimate which leads to

$$\|e^{(t-s)\Delta}\mathcal{F}(z)(s)\|_{L^\infty(\mathbb{R}^{d+2})} \leq C_1 \| z \|_{\mathcal{X}^T},$$

provided that $t - s > 0$. Next, we take $t \in (0,T]$ arbitrarily, and we derive from (8) and the above estimate that

$$\left|\mathbb{T}(z)(t) - e^{t\Delta}(z_0)\right| = \left|\int_0^t e^{(t-s)\Delta}\mathcal{F}(z)(s)ds\right| \leq \int_0^t \left|e^{(t-s)\Delta}\mathcal{F}(z)(s)\right|ds$$

$$\leq \int_0^t \| \mathcal{F}(z)(s) \|_{L^\infty(\mathbb{R}^{d+2})} ds \leq C_1 T(\| z \|_{\mathcal{X}^T}).$$

Taking the supremum on $(0,T]$, we obtain

$$\sup_{t\in(0,T]} \left\|\mathbb{T}(z) - e^{t\Delta}(z_0)\right\|_{L^\infty(R^{d+2})} \leq C_1 T \| z \|_{\mathcal{X}^T}. \tag{9}$$

On the other hand, with $t \in (0,T]$ and $x \in \mathbb{R}^{d+2}$, we apply ∇ to (2) with $f = z_0$ which leads the following identity

$$\begin{aligned}\nabla_x e^{t\Delta}(z_0)(x) &= \frac{1}{(4\pi t)^{\frac{d+2}{2}}} \int_{\mathbb{R}^{d+2}} \nabla_x e^{-\frac{|x-y|^2}{4t}} z_0(y)dy \\ &= \frac{1}{(4\pi t)^{\frac{d+2}{2}}} \int_{\mathbb{R}^{d+2}} \left(-\nabla_y e^{-\frac{|x-y|^2}{4t}}\right) z_0(y)dy.\end{aligned}$$

Since $z_0 \in W^{1,\infty}(\mathbb{R}^{d+2})$ we derive from the integration by parts that

$$\nabla_x e^{t\Delta}(z_0) = \frac{1}{(4\pi t)^{\frac{d+2}{2}}} \int_{\mathbb{R}^{d+2}} e^{-\frac{|x-y|^2}{4t}} \nabla z_0(y)dy = e^{t\Delta}\nabla z_0.$$

Taking L^{∞}-estimate with (3), we have

$$\|\nabla_x e^{t\Delta} z_0(t)\|_{L^{\infty}(\mathbb{R}^{d+2})} \leq \|\nabla z_0\|_{L^{\infty}(\mathbb{R}^{d+2})}, \forall t \in (0,T].$$

and

$$\|e^{t\Delta} z_0\|_{L^{\infty}(\mathbb{R}^{d+2})} \leq \|z_0\|_{L^{\infty}(\mathbb{R}^{d+2})}, \forall t \in (0,T].$$

Thus, we obtain

$$\|e^{t\Delta} z_0(t)\|_{\mathcal{X}^T} \leq \|z_0\|_{W^{1,\infty}(\mathbb{R}^{d+2})}. \tag{10}$$

Considering $\varepsilon > 0$ small enough such that $t-\varepsilon > 0$, and estimate (2), we derive

$$\begin{aligned}
&\nabla_x \int_0^{t-\varepsilon} e^{(t-s)\Delta \mathcal{F}(z)(s) ds} \\
&= \int_0^{t-\varepsilon} \frac{1}{(4\pi(t-s))^{\frac{d+2}{2}}} \int_{\mathbb{R}^{d+2}} \nabla_x e^{-\frac{|x-y|^2}{4(t-s)}} \mathcal{F}(z)(y,s) ds \\
&= \int_0^{t-\varepsilon} \frac{1}{\sqrt{t-s}} \int_{\mathbb{R}^{d+2}} \frac{1}{(4\pi(t-s))^{\frac{d+2}{2}} e^{-\frac{|x-y|^2}{8(t-s)}} \mathcal{F}(z)(y,s)} \\
&\times \left(-\frac{(x-y)}{2\sqrt{(t-s)}} e^{-\frac{4x-y|^2}{8(t-s)}}\right) dyds.
\end{aligned}$$

Note that, we can estimate as follows

$$\left|-\frac{(x-y)}{2\sqrt{(t-s)}} e^{-\frac{|x-y|^2}{8(t-s)}}\right| \leq C, \forall x,y \in \mathbb{R}^{d+2}, t-s > 0,$$

where the constant C doesn't depend on the other parameters. Thus, we obtain

$$\left|\nabla_x \int_0^{t-\varepsilon} e^{(t-s)\Delta} \mathcal{F}(z)(s) ds\right| \leq C \int_0^{t-\varepsilon} \frac{1}{\sqrt{t-s}} \left|\int_{R^{d+2}} \frac{1}{\left(4\pi(t-s)\right)^{\frac{d+2}{2}}} e^{-\frac{|x-y|^2}{8(t-s)}} |\mathcal{F}(z)(y,s)| dyds\right|$$

$$\leq C_2 \int_0^{t-\varepsilon} \frac{1}{\sqrt{t-s}} |\mathcal{F}(z)(s)|_{L^{\infty}(R^{d+2})} ds \leq \int_0^{t-\varepsilon} \frac{1}{\sqrt{t-s}} |z|_{\mathcal{X}^T} ds \leq C_3 T \|z\|_{\mathcal{X}^T}.$$

Taking $\varepsilon \to 0$ and (8), get

$$\sup_{t\in(0,T]} \|\nabla \mathbb{T}(z)(t) - \nabla e^{t\Delta} z_0\|_{L^{\infty}(\mathbb{R}^{d+2})} \leq C_3 T\left(\| z \|_{\mathcal{X}^T}\right). \tag{11}$$

Thanks to (9) và (11), we conclude the following

$$\|\mathbb{T}(z) - e^{t\Delta} z_0\|_{\mathcal{X}^T} \leq (C_1 + C_3) T \| z \|_{\mathcal{X}^T}.$$

Finally, we choose $T \leq \frac{1}{C_1 + C_3}$ which leads to

$$\|\mathbb{T}(z) - e^{t\Delta} z_0\|_{\mathcal{X}^T} \leq \| z \|_{\mathcal{X}^T}.$$

Moreover, we introduce

$$G(x,t) = \frac{1}{(4\pi t)^{\frac{d+2}{2}}} e^{-\frac{|x|^2}{4t}}, t > 0,$$

which is radially symmetric, and we rewrite

$$\mathbb{T}(z) = G(t) * z_0 + \int_0^t G(t-s) * \mathcal{F}(z)(s) ds,$$

where the notation '*' stands for the convolution product. It is easy to see that the convolution product of two radially symmetric functions remains radially symmetric. So, we derive $\mathbb{T}(z)$ if $z \in \mathbb{B}_0$.

- *The proof of* $(H2)$ *:* Considering $z_1, z_2 \in \mathbb{B}_0$, then, we can write

$$\mathbb{T}(z_1) - \mathbb{T}(z_2) = \int_0^t e^{(t-s)\Delta}\big(\mathcal{F}(z_1) - \mathcal{F}(z_2)\big)(s)ds.$$

Regarding (7) and the facts that $\|z_1\|_{\mathcal{X}^T} \le 1$ and $\|z_1\|_{\mathcal{X}_T} \le 1$, we quickly obtain

$$\begin{aligned}&|\mathcal{F}(z_1)(y,s) - \mathcal{F}(z_2)(y,s)|\\ &\le C_4(|z_1(y,s) - z_2(y,s)| + |\nabla z_1(y,s) - \nabla z_2(y,s)|)\\ &\le C_4\|z_1(s) - z_2(s)\|_{W^{1,\infty}(\mathbb{R}^{d+2})} \le C_4\|z_1 - z_2\|_{\mathcal{X}^T}.\end{aligned}$$

In particular, we derive

$$\|\mathcal{F}(z_1)(s) - \mathcal{F}(z_2)(s)\|_{L^\infty(\mathbb{R}^{d+2})} \le C_4\|z_1 - z_2\|_{\mathcal{X}^T}.$$

Using the above estimate and (2), we have the following estimate

$$\begin{aligned}&\|\mathbb{T}(z_1)(t) - \mathbb{T}(z_2)(t)\|_{L^\infty(\mathbb{R}^{d+2})}\\ &\le C_4\int_0^t \|z_1 - z_2\|_{\mathcal{X}^T}ds \le C_4T\|z_1 - z_2\|_{\mathcal{X}^T}, t \in (0,T].\end{aligned} \tag{12}$$

Similarly to (11), we also obtain

$$\|\nabla\mathbb{T}(z_1)(t) - \nabla\mathbb{T}(z_2)(t)\|_{L^\infty(\mathbb{R}^{d+2})} \le C_5T\|z_1 - z_2\|_{\mathcal{X}^T}, t \in (0,T]. \tag{13}$$

Combining (12) and (13), we conclude

$$\|\mathbb{T}(z_1) - \mathbb{T}(z_2)\|_{\mathcal{X}^T} \le (C_4 + C_5)T\|z_1 - z_2\|_{\mathcal{X}^T}.$$

Taking $T \le \frac{1}{2(C_4+C_5)}$, thus, $\lambda = (C_4 + C_5)T < 1$ and the following holds

$$\|\mathbb{T}(z_1) - \mathbb{T}(z_2)\|_{\mathcal{X}^T} \le \lambda\|z_1 - z_2\|_{\mathcal{X}^T}.$$

Finally, we conclude (H2).

From the fact that $W^{1,\infty}(\mathbb{R}^{d+2}) \cap L^\infty_{\text{rad}}(\mathbb{R}^{d+2})$ is a Banach space, we apply the Banach fixed point theorem to derive the existence and uniqueness of the fixed point $z \in \mathbb{B}_0$ of $\mathbb{T}$ such that

$$z(t) = \mathbb{T}(z)(t), \forall t \in (0,T].$$

In particular, by the parabolic regularity, we have $z(t) \in C^2(\mathbb{R}^{d+2})$ and it satisfies equation (8) for all $(x,t) \in \mathbb{R}^{d+2} \times (0,T)$ pointwise. In addition, since $z \in \mathbb{B}_0$, we get

$$\| z \|_{\mathcal{X}^T} = \| \mathbb{T}(z) \|_{\mathcal{X}^T} \le \|z_0\|_{W^{1,\infty}(\mathbb{R}^{d+2})} + 1, t \in (0,T],$$

and it holds when $t = 0$. Finally, we get the conclusion of the Proposition.

Consequently, Proposition 2.1 implies the following result:

Proposition 2. 2: *Let $d \ge 1$ be an integer number and initial choice*

$$u_0 \in W^\infty_{1+r^2}(\mathbb{R}_+).$$

Then, there exists $T = T(u_0) > 0$ such that problem (1) has a unique solution on $[0,T]$ and $u(t) \in W^{1,\infty}_{1+r^2}(\mathbb{R}_+), \forall t \in [0,T]$. In particular, we have the following estimate

$$|u(r,t)| + |\partial_r u(r,t)| \le \frac{C(u_0)}{1+r^2}, \forall (r,t) \in (0,+\infty) \times [0,T].$$

Proof: Let $u_0 \in W^{1,\infty}(\mathbb{R}_+)$, then, it follows that $z_0 \in W^{1,\infty}(\mathbb{R}^{d+2}) \cap L^\infty_{\text{rad}}(\mathbb{R}^{d+2})$. Applying **Proposition 2.1**, we obtain the existence and the uniqueness of z on $[0,T(z_0)] = [0,T(u_0)]$ and $z(t) \in C^2(\mathbb{R}^{d+2})$, then, the problem (1) and (8) are equivalent, this leads to the existence and the uniqueness of u, the solution to equation (1) on $[0,T]$. In particular, the estimate involving to u immediately follows the following

$$\| z(t) \|_{W^{1,\infty}(\mathbb{R}^{d+2})} \le \|z_0\|_{W^{1,\infty}(\mathbb{R}^{d+2})} + 1, \text{ for all } t \in (0,T].$$

Finally, we get the conclusion of the Proposition.

4. Conclusion

As we showed in Proposition 2.2, the Cauchy problem for (1) is totally solved in $W^{1,\infty}_{1+r^2}(\mathbb{R}_+)$ a subspace of $L^\infty(\mathbb{R}_+)$. In comparison to the result proved by [5] where the authors considered the problem in $C^\infty_0(\mathbb{R}_+)$, our result is better. The proof technique replies to the idea given by Biernat and Seki, 2019 in [10] that extends the problem to a similar issue that $\mathbb{R}^{d+2}-$ dimensional one, and then, we reply on the theory of semi-group $e^{t\Delta}, t > 0$ to get the conclusion. The new contribution is to use the suitable transformation

$$z = (1 + |x|^2)\tilde{u}$$

which avoids the unbounded property of the nonlinear term r^2u^3 as $r \to +\infty$. Inspired by the current result, we suggest an open question if the hypothesis $W^{1,\infty}_{1+r^2}(\mathbb{R}_+)$ can be replaced by $L^{\infty}_{1+r^2}(\mathbb{R}_+)$. In the future, we are interested in solving the open one that leads to new tools and techniques discovered.

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